02.10.2020

GTLA Lecture ①

Let $\langle , \rangle : V \times V \to \mathbb{F}$ be a sesquilinear form.

Let W be a finite dimensional subspace of V with $W \cap W^{\perp} = 0$.

Let $f: W \to W$ be a linear transformation.

Let $f^*: W \to W$ be the adjoint linear transformation.

(a) f is <u>self adjoint</u> if $f = f^*$.

(b) f is an <u>isometry</u> if $ff^* = 1$.

(c) f is <u>normal</u> if $ff^* = f^*f$

The favorite example is $W = \mathbb{C}^n$ with the standard dot product.

Let A be the matrix of f and

A the matrix of f^*

with respect to the favorite basis $\{e_1, \ldots, e_n\}$.

$$f: \mathbb{C}^n \to \mathbb{C}^n \quad v \mapsto Av \qquad \text{and} \qquad f^*: \mathbb{C}^n \to \mathbb{C}^n \quad v \mapsto A^* v.$$

Then

$$A^* = \bar{A}^t.$$

and

A is <u>self-adjoint</u> if $A = A^*$

A is <u>unitary</u> if $AA^* = 1$

A is <u>normal</u> if $AA^* = A^*A$.

02.10.2020

GTLA Lecture 2

<u>Proposition</u> Let $A \in M_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$ and let $V_\lambda = \ker(\lambda - A)$

If $AA^* = A^*A$ then

(a) V_λ is A-invariant, (b) V_λ is A^*-invariant

(c) $V_\lambda^\perp$ is A-invariant, (d) $V_\lambda^\perp$ is A^*-invariant.

<u>Proof</u> (a) To show: If $p \in V_\lambda$ then $Ap \in V_\lambda$.

Assume $p \in V_\lambda$.

Then $(\lambda - A)p = 0$.

So $Ap = \lambda p$.

So $Ap \in V_\lambda$.

So V_λ is A-invariant.

(b) To show: If $p \in V_\lambda$ then $A^* p \in V_\lambda$.

Assume $p \in V_\lambda$.

~~Case 1~~ $\cancel{\lambda \neq 0}$. ~~Then~~ $\cancel{p = \frac{1}{\lambda} Ap}$.

~~So~~ $\cancel{A^* p = A^*\left(\frac{1}{\lambda} Ap\right) = \frac{1}{\lambda} A^* A p = \frac{1}{\lambda} A A^* p}$

To show: $A^* p \in V_\lambda$

To show: $AA^* p = \lambda A^* p$.

$$AA^* p = A^* A p = A^* \lambda p = \lambda A^* p.$$

So $A^* p \in V_\lambda$.

(d) To show: $V_\lambda^\perp$ is A^* invariant.

To show: If $v \in V_\lambda^\perp$ then $A^* v \in V_\lambda^\perp$

Assume $v \in V_\lambda^\perp$.

To show: If $p \in V_\lambda$ then $\langle p, A^* v \rangle = 0$.

Assume $p \in V_\lambda^\perp$.

To show: $\langle A^* v, p \rangle = 0$

$$\langle A^* v, p \rangle = \overline{\langle p, A^* v \rangle} = \overline{\langle Ap, v \rangle} = \overline{\langle \lambda p, v \rangle}$$

$$= \overline{\lambda \langle p, v \rangle} = \bar{\lambda} \cdot \bar{0} = 0.$$

(c) To show: $V_\lambda^\perp$ is A-invariant.

To show: If $v \in V_\lambda^\perp$ then $Av \in V_\lambda^\perp$.

Assume $v \in V_\lambda^\perp$

To show: If $p \in V_\lambda$ then $\langle Av, p \rangle = 0$.

Assume $p \in V_\lambda$.

To show: $\langle Av, p \rangle = 0$

$\langle Av, p \rangle = \langle v, A^* p \rangle = 0$ since $A^* p \in V_\lambda$ and $v \in V_\lambda^\perp$. □

<u>Theorem</u> (Spectral theorem)

Let $n \in \mathbb{Z}_{>0}$ and $V = \mathbb{C}^n$ with the standard dot product.

(a) Let $A \in M_n(\mathbb{C})$ such that $AA^* = A^*A$. Then there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and a unitary matrix $U \in M_n(\mathbb{C})$ such that

$$U^{-1}AU = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

(b) Let $f: V \to V$ be a normal linear transformation. Then there exists an orthonormal basis $(u_1, u_2, \ldots, u_n)$ of V consisting of eigenvectors of f.

<u>Proposition</u> ~~Let~~ The <u>unitary</u> group is

$$U_n = \{ A \in M_n(\mathbb{C}) \mid AA^* = 1 \}$$

There is a bijection

$$\left\{ \begin{matrix} \text{ordered orthonormal} \\ \text{bases } (u_1, \ldots, u_n) \end{matrix} \right\} \longrightarrow U_n(\mathbb{C})$$

$$(u_1, \ldots, u_n) \longmapsto \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{pmatrix}$$

Cor If A is self adjoint then A has real eigenvalues.

Proof If A is self adjoint then

$$AA^* = AA = A^*A$$ and A is normal.

Let $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = U^{-1}AU$

Then $\overline{D} = (U^{-1}AU)^* = U^*AU = U^*A^*U = U^*AU = D$.

So $\overline{\lambda_i} = \lambda_i$. So $\lambda_i \in \mathbb{R}$. //

Corollary If A is unitary then the eigenvalues of A have absolute value 1.

Proof Assume A is unitary.

Then $AA^* = 1 = A^*A$ and A is normal.

Let $D = U^{-1}AU = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$.

Then $$DD^* = (U^{-1}AU)(U^{-1}AU)^*$$
$$= U^{-1}AUU^*A^*U = U^{-1}AA^*U = 1.$$

So $\lambda_i \overline{\lambda_i} = 1$. So $|\lambda_i|^2 = 1$. //