13.10.2020
GTLA Lecture (1)

Let $V = \mathbb{C}^n$ with the standard dot product,

$$\left\langle \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \right\rangle = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1\bar{v}_1 + \cdots + u_n\bar{v}_n$$

Let $f: V \to V$ be a linear transformation.
Let $f^*: V \to V$ be the adjoint linear transformation,
i.e. if $u, v \in V$ then $\langle f(u), v\rangle = \langle u, f^*(v)\rangle$.
Then

(a) f is <u>self adjoint</u> if $f = f^*$,

(b) f is an <u>isometry</u> if $ff^* = 1$.

(c) f is <u>normal</u> if $ff^* = f^*f$.

Let A be the matrix of f and
A^* the matrix of f^*
with respect to the favorite basis
$(e_1, e_2, \ldots, e_n)$

$$\begin{aligned} f: V &\to V \\ v &\mapsto Av \end{aligned} \quad \text{and} \quad \begin{aligned} f^*: V &\to V \\ v &\mapsto A^*v \end{aligned}$$

Then

$$A^* = \bar{A}^t.$$

Then

A is <u>self adjoint</u>, or <u>Hermitian</u>, if $A = A^*$,

A is <u>unitary</u> if $AA^* = 1$,

A is <u>normal</u> if $AA^* = A^*A$.

The <u>unitary group</u> is

$$U_n(\mathbb{C}) = \{ A \in M_n(\mathbb{C}) \mid AA^* = 1 \}.$$

There is a bijection

$$\left\{ \begin{array}{l} \text{ordered orthonormal} \\ \text{bases } (u_1, \ldots, u_n) \end{array} \right\} \longrightarrow U_n(\mathbb{C})$$

$$(u_1, \ldots, u_n) \longmapsto \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{pmatrix}$$

<u>Proposition</u> Let $A \in M_n(\mathbb{C})$

Let $\lambda \in \mathbb{C}$ and $V_\lambda = \ker(\lambda - A)$

If $AA^* = A^*A$ then

(1) V_λ is A-invariant, (2) V_λ is A^*-invariant,

(3) $V_\lambda^\perp$ is A-invariant, (4) $V_\lambda^\perp$ is A^*-invariant

Theorem (Spectral Theorem)

Let $n \in \mathbb{Z}_{>0}$ and $V = \mathbb{C}^n$ with the standard dot product.

(a) Let $A \in M_n(\mathbb{C})$ such that $AA^* = A^*A$. Then there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and a unitary matrix $U \in M_n(\mathbb{C})$ such that

$$U^{-1} A U = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

(b) Let $f: V \to V$ be a normal linear transformation. Then there exists an orthonormal basis $(u_1, u_2, \ldots, u_n)$ of V consisting of eigenvectors of f.

Proof

Proof by induction on n.

Base case $n=1$: $A \in M_1(\mathbb{C})$. So $A = (\lambda_1)$ with $\lambda_1 \in \mathbb{C}$ and $U = (1)$.

Induction step:

Let $\lambda \in \mathbb{C}$ be a root of $\det(x - A)$.

Let $V_\lambda = \ker(\lambda - A)$, the λ-eigenspace of A.

Since $\det(\lambda - A) = 0$ then $V_\lambda = \ker(\lambda - A) \neq 0$

Let $k = \dim(V_\lambda)$ and let

$(u_1, u_2, \ldots, u_k)$ be an orthonormal basis of V_λ

(construct it with Gram-Schmidt). Then

$(u_1, u_2, \ldots, u_k)$ are all eigenvectors of A of eigenvalue λ.

Since A is normal then $V_\lambda^\perp$ is A-invariant and $V_\lambda^\perp$ is A^*-invariant.

Let

$$g : V_\lambda^\perp \to V_\lambda^\perp, \quad v \mapsto Av \qquad \text{and} \qquad g^* : V_\lambda^\perp \to V_\lambda^\perp, \quad v \mapsto A^* v.$$

Then $gg^* = g^*g$ and so g is normal.

By induction, there exists an orthonormal basis $(u_{k+1}, \ldots, u_n)$ of eigenvectors of g.

Since $(u_{k+1}, \ldots, u_n)$ are eigenvectors of g they are eigenvectors of A.

Since $(u_{k+1}, \ldots, u_n)$ are in $V_\lambda^\perp$ they are orthogonal to $(u_1, u_2, \ldots, u_k)$.

So $\{u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_n\}$ is an orthonormal basis of eigenvectors of $V = V_\lambda \oplus V_\lambda^\perp$. //

Corollary If A is unitary then the eigenvalues of A have absolute value 1.

Proof Assume A is unitary.

Then $AA^* = 1 = A^*A$ and A is normal.

Let $D = U^{-1}AU = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

Then

$$DD^* = (U^{-1}AU)(U^{-1}AU)^* = (U^{-1}AU)(U^*A^*U)$$
$$= U^{-1}AA^*U = U^{-1}U = 1.$$

So $\lambda_i \bar{\lambda}_i = 1$.

So $|\lambda_i|^2 = 1$.

So $|\lambda_i| = 1$. //