Cyclic and dihedral groups

Let $r, s \in M_n(\mathbb{C})$ be given by

$$r(i,j) = \begin{cases} 1, & \text{if } j = i-1 \bmod n, \\ 0, & \text{otherwise,} \end{cases} \qquad s(i,j) = \begin{cases} 1, & \text{if } j = n-i \\ 0, & \text{otherwise.} \end{cases}$$

so that

$$r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{if } n = 2,$$

$$r = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{if } n = 3,$$

$$r = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{if } n = 4,$$

$$r = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{if } n = 5.$$

The <u>cyclic group</u> $\mathbb{Z}_n$ is $\mathbb{Z}_n = \{1, r, r^2, \dots, r^{n-1}\}$

The <u>dihedral group</u> is

$$D_n = \left\{ \begin{matrix} 1, r, r^2, \dots, r^{n-1} \\ s, sr, sr^2, \dots, sr^{n-1} \end{matrix} \right\}$$

so that D_n is presented by generators r, s with relations

$$r^n = 1, \quad s^2 = 1, \quad rs = sr^{-1}.$$

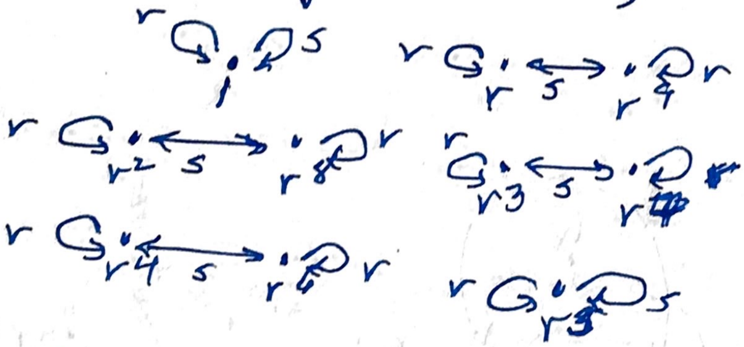
The conjugation action of D_n on itself is

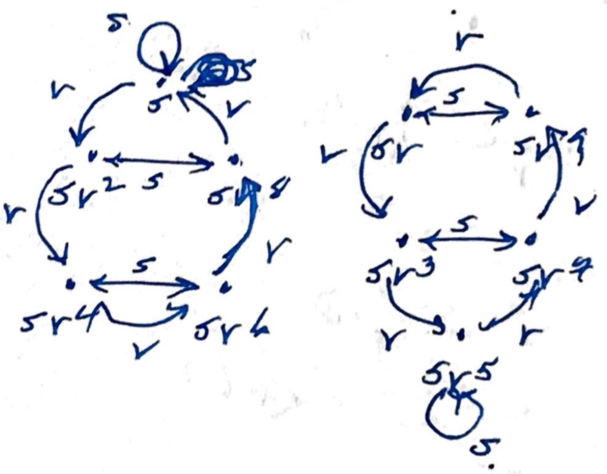
$$D_n \times D_n \to D_n$$
$$(g, x) \mapsto g \diamond x \quad \text{with } g \diamond x = g x g^{-1}.$$

Explicitly,

$$r \diamond r^k = r r^k r^{-1} = r^k$$
$$s \diamond r^k = s r^k s^{-1} = s r^k s = s s r^{-k} = r^{-k} = r^{n-k}$$
$$r \diamond s r^k = r s r^k r^{-1} = s r^{-1} r^{k-1} = s r^{k-2}$$
$$s \diamond s r^k = s s r^k s^{-1} = r^k s = s r^{-k} = s r^{n-k}$$

Pictorially, for $n=10$,





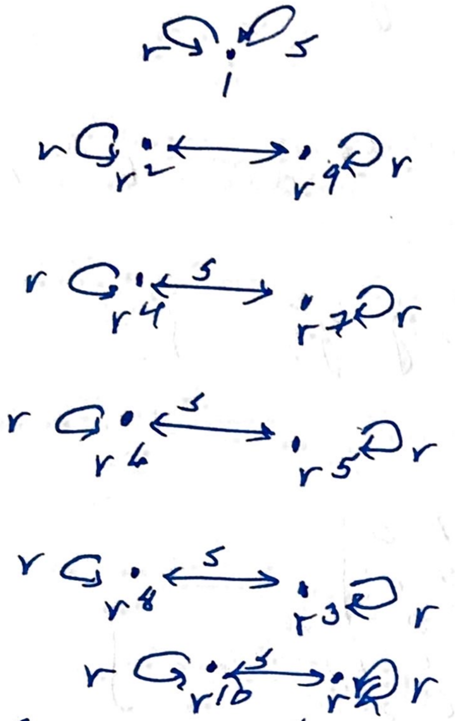
Conjugacy classes:

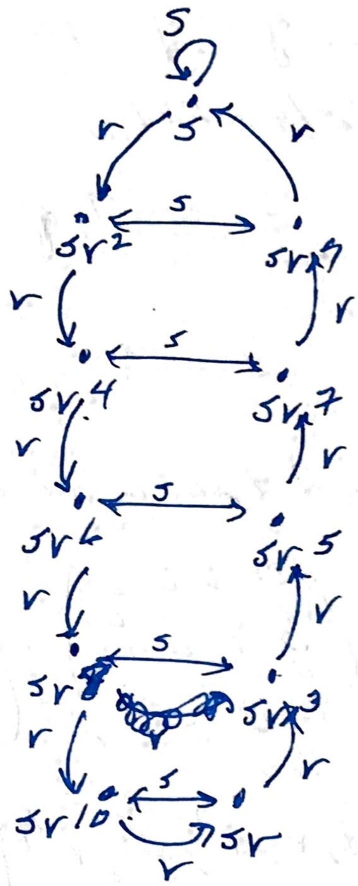
$\{1\}$ $\quad \{r, r^9\}$ $\quad \{s, sr^2, sr^4, sr^6, sr^8\}$

$\{r^2, r^8\}$ $\quad \{r^3, r^7\}$ $\quad \{sr, sr^3, sr^5, sr^7, sr^9\}$

$\{r^4, r^6\}$ $\quad \{r^5\}$

Pictorially, for $n=11$,





Conjugacy classes:

$\{1\}$

$\{r^2, r^9\}$

$\{r^4, r^7\}$

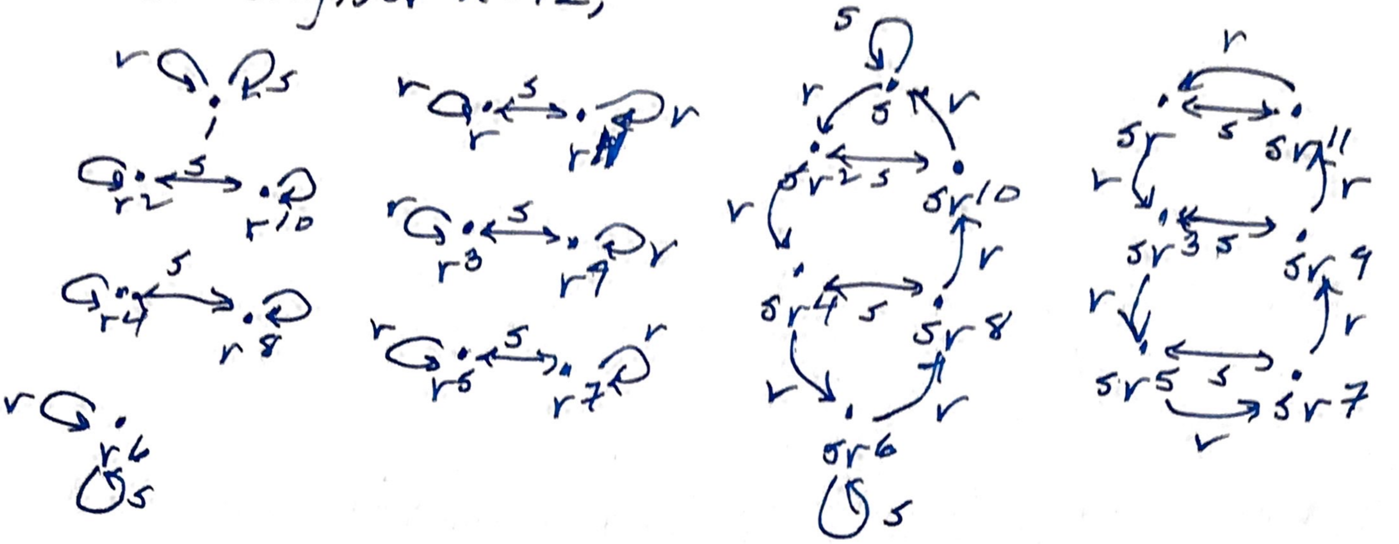
$\{r^6, r^5\}$

$\{r^8, r^3\}$

$\{r^{10}, r\}$

and $\left\{\begin{matrix} s, sr^2, sr^4, sr^6, sr^8, sr^{10}, \\ sr, sr^3, sr^5, sr^7, sr^9 \end{matrix}\right\}$

Pictorially, for $n=12$,



Conjugacy classes:

$\{1\}$

$\{r^2, r^{10}\}$ $\{r, r^{11}\}$ $\{s, sr^2, sr^4, sr^6, sr^8, sr^{10}\}$

$\{r^4, r^8\}$ $\{r^3, r^9\}$ $\{sr, sr^3, sr^5, sr^7, sr^9, sr^{11}\}$.

$\{r^6\}$ $\{r^5, r^7\}$

<u>The group μ_4</u> $\mu_4 = \{1, i, i^2, i^3\}$ 4^{th} roots of unity.

$= \{1, i, -1, -i\}$

$\mathbb{C} = \{a + bi + ci^2 + di^3 \mid a, b, c, d \in \mathbb{R}, i^2 = -1\}$

$= \{x + yi \mid x, y \in \mathbb{R}, i^2 = -1\}$.

Counting orbits using fixed points

Proposition Let S be a finite set. Let G be a group acting on S. Then

$$\begin{pmatrix}\#\text{ of}\\ \text{orbits}\end{pmatrix}(\mathrm{Card}(G)) = \sum_{g\in G}\begin{pmatrix}\#\text{ of fixed points}\\ \text{of } g \text{ on } S\end{pmatrix}$$

Proof Let $Gx_1, \ldots, Gx_N$ be the distinct orbits.

$$\sum_{g\in G}\begin{pmatrix}\#\text{ of fixed points}\\ \text{of } g \text{ on } S\end{pmatrix} = \sum_{g\in G} \mathrm{Card}\{x\in S \mid gx = x\}$$

$$= \mathrm{Card}\{(g,x) \mid g\in G, x\in S \text{ and } gx = x\}$$

$$= \sum_{x\in S} \mathrm{Card}\{g\in G \mid gx = x\}$$

$$= \sum_{i=1}^{N} \sum_{x\in Gx_i} \mathrm{Card}(\mathrm{Stab}_G(x))$$

$$= \sum_{i=1}^{N} \sum_{x\in Gx_i} \mathrm{Card}(\mathrm{Stab}_G(x_i))$$

$$= \sum_{i=1}^{N} \mathrm{Card}(Gx_i)\,\mathrm{Card}(\mathrm{Stab}_G(x_i))$$

$$= \sum_{i=1}^{N} \mathrm{Card}(G) = N\cdot \mathrm{Card}(G)$$

$$= \begin{pmatrix}\#\text{ of}\\ \text{orbits}\end{pmatrix}\cdot \mathrm{Card}(G) \quad //.$$