Let G be a group.

Let $\mathcal{S}$ be the set of subsets of G

The <u>conjugation</u> <u>action</u> of G on $\mathcal{S}$ is

$$G \times \mathcal{S} \longrightarrow \mathcal{S}$$
$$(g, S) \longmapsto g \diamond S \quad \text{where } g \diamond S = gSg^{-1}$$

with $gSg^{-1} = \{gsg^{-1} \mid s \in S\}$.

<u>Example</u> $D_2 = \{1, r, s, sr\}$.

$$\mathcal{S} = \left\{ \begin{array}{c} \phi \\ \{1\} \quad \{r\} \quad \{s\} \quad \{sr\} \\ \{1,r\} \quad \{1,s\} \quad \{1,sr\} \quad \{r,s\} \quad \{r,sr\} \quad \{s,sr\} \\ \{1,r,s\} \quad \{1,r,sr\} \quad \{1,s,sr\} \quad \{r,s,sr\} \\ \{1,r,s,sr\} \end{array} \right\}$$

and $r \diamond \{1, r, sr\} = \{r1r^{-1}, r(r)r^{-1}, r(sr)r^{-1}\}$
$= \{1, r, sr\}$.

Let S be a subset of G.

The <u>normalizer of S</u> in G is

$$N(S) = \{x \in G \mid xSx^{-1} = S\} = \mathrm{Stab}_G(S)$$

<u>Proposition</u> Let H be a subgroup of G.

(a) H is a normal subgroup of $N(H)$.

(b) If K is a subgroup of G and H is a normal subgroup of K then $K \subseteq N(H)$.

<u>Proof</u> (a) To show: If $h \in H$ and $n \in N(H)$ then $nhn^{-1} \in H$.

Assume $h \in H$ and $n \in N(H)$.

Since $n \in N(H)$ then $nHn^{-1} = H$.

So $nhn^{-1} \in H$.

(a') To show: If $h \in H$ then $h \in N(H)$.

Assume $h \in H$

To show: $hHh^{-1} = H$.

Since H is a subgroup, it is closed under multiplication and $hHh^{-1} \subseteq H$.

Since $\begin{array}{rcl} H & \to & hHh^{-1} \\ x & \mapsto & hxh^{-1} \end{array}$ has inverse $\begin{array}{rcl} hHh^{-1} & \to & H \\ y & \mapsto & h^{-1}yh \end{array}$

then $\mathrm{Card}(H) = \mathrm{Card}(hHh^{-1})$.

So $hHh^{-1} = H$.

So $h \in N(H)$.

(b) Assume K is a subgroup of G and
H is a normal subgroup of K.

To show: $K \subseteq N(H)$.

To show: If $k \in K$ then $k \in N(H)$.

Assume $k \in K$.

To show: $k \in N(H)$.

To show: $kHk^{-1} = H$.

Since H is a normal subgroup of K
then $kHk^{-1} = H$. //

The Sylow theorems

Let G be a finite group. Let $p \in \mathbb{Z}_{>0}$ be prime.

Write
$$\mathrm{Card}(G) = p^a b, \text{ where } b \text{ is not divisible by } p.$$

The set of <u>p-Sylow subgroups of G</u> is

$$\mathcal{S} = \{ Q \mid Q \text{ is a subgroup and } \mathrm{Card}(Q) = p^a \}$$

The group G acts on $\mathcal{S}$ by conjugation

$$\begin{aligned} G \times \mathcal{S} &\longrightarrow \mathcal{S} \\ (g, Q) &\longmapsto gQg^{-1} \end{aligned}$$

Then (1) $\mathcal{S} \neq \emptyset$

(2) The action of G on $\mathcal{S}$ by conjugation has only one orbit

(3) $\mathrm{Card}(\mathcal{S}) = 1 \bmod p$

(4) $\mathrm{Card}(\mathcal{S})$ divides $\mathrm{Card}(G)$.

Let G be a group of order 84.

Let n_p be the number of p-Sylow subgroups of G

$$84 = 2^2 \cdot 3 \cdot 7.$$

So n_2 divides 84 and $n_2 \equiv 1 \mod 2$.

So n_2 is 1 or 3 or 7 or 21.

Then n_3 divides 84 and $n_3 \equiv 1 \mod 3$.

So $n_3 = 1$ or 4, or 7 or ~~11~~ or 14 or ~~17~~ or ~~22~~ or ~~[illegible] or 29 or 33 or~~ ~~23~~ or ~~26~~ or ~~29~~ or ~~32~~ or ~~35~~ or ~~38~~ or ~~41~~ or ~~45~~

Then n_7 divides 84 and $n_7 \equiv 1 \mod 7$

So $n_7 = 1$ or ~~8~~ or ~~15~~ or ~~23~~ or ~~30~~ or ~~37~~ or ~~44~~

So G has a normal subgroup K with $\mathrm{Card}(K) = 7$.

and G/K is a group of order 12.