Let $\mathbb{A}$ be a commutative ring.

An <u>ideal</u>, or <u>submodule</u>, of $\mathbb{A}$ is $M \subseteq \mathbb{A}$ such that

(1) If $m_1, m_2 \in M$ then $m_1 + m_2 \in M$,

(2) If $m \in M$ and $a \in \mathbb{A}$ then $am \in M$.

$\mathbb{A}$ is a <u>PID</u>, or principal ideal domain if $\mathbb{A}$ satisfies

(1) If $a, b, c \in \mathbb{A}$ and $c \neq 0$ and $ac = bc$ then $a = b$,

(2) If M is an ideal of $\mathbb{A}$ then there exists $\ell \in \mathbb{A}$ such that $M = \ell\mathbb{A}$.

Let $\mathbb{F}$ be a field. We showed:

<u>Proposition</u> (Euclidean algorithm for $\mathbb{F}[x]$)
Let $a(x), b(x) \in \mathbb{F}[x]$ with $b(x)$ monic. Then there exist unique $q(x), r(x) \in \mathbb{F}[x]$ such that

$$a(x) = q(x)b(x) + r(x) \quad \text{and}$$

$$\deg(r(x)) < \deg(b(x)).$$

<u>Theorem</u> $\mathbb{F}[x]$ is a PID.

<u>Proof</u> To show: (a) $\mathbb{F}[x]$ satisfies the cancellation law.

(b) If M is an ideal then there exists $\ell(x) \in \mathbb{F}[x]$ such that $M = \ell(x)\mathbb{F}[x]$.

(b) Let M be an ideal of $\mathbb{F}[x]$.

Let $m(x) \in M$ be such that

if $p(x) \in M$ then $\deg(p(x)) \geq \deg(m(x))$.

Let $m(x) = m_0 + m_1 x + \cdots + m_d x^d$ with $m_d \neq 0$

Let $\ell(x) = \frac{1}{m_d} m(x)$.

Since $\frac{1}{m_d} \in \mathbb{F}[x]$ and $m(x) \in M$ then $\ell(x) \in M$.

To show: $\ell(x)\mathbb{F}[x] = M$.

To show: (ba) $\ell(x)\mathbb{F}[x] \subseteq M$.

(bb) $M \subseteq \ell(x)\mathbb{F}[x]$.

(ba) Assume $p(x) \in \mathbb{F}[x]$.

To show: $\ell(x)p(x) \in M$.

Since $\ell(x) \in M$ and M is an ideal

then $\ell(x)p(x) \in M$.

So $\ell(x)\mathbb{F}[x] \subseteq M$.

(bb) To show: $M \subseteq \ell(x)\mathbb{F}[x]$.

To show: If $a(x) \in M$ then $a(x) \in \ell(x)\mathbb{F}[x]$.

Assume $a(x) \in M$.

By the Euclidean algorithm for $\mathbb{F}[x]$ there exist $r(x), q(x) \in \mathbb{F}[x]$ such that

$$a(x) = q(x)\ell(x) + r(x) \quad \text{and} \quad \deg(r(x)) < \deg(\ell(x))$$

Since M is an ideal and $a(x), \ell(x) \in M$ then

$$r(x) = a(x) - q(x)\ell(x) \in M.$$

Since $\deg(r(x)) < \deg(\ell(x))$ and $\ell(x)$ has minimal degree among elements of M then $r(x) = 0$.

So $a(x) = q(x)\ell(x) \in \ell(x)\mathbb{F}[x]$.

So $M \subseteq \ell(x)\mathbb{F}[x]$.

So $M = \ell(x)\mathbb{F}[x]$.

(a) To show: $\mathbb{F}[x]$ satisfies the cancellation law.

To show: If $b(x), c(x) \in \mathbb{F}[x]$ and $c(x) \neq 0$ and $b(x)c(x) = 0$ then $b(x) = 0$.

Assume $b(x), c(x) \in \mathbb{F}[x]$ and $c(x) \neq 0$ and $b(x)c(x) = 0$.

Proof of contrapositive:

Write $b(x) = b_0 + b_1x + \cdots + b_mx^m$ with $b_m \neq 0$

and $c(x) = c_0 + c_1x + \cdots + c_nx^n$ with $c_n \neq 0$.

Then

$$b(x)c(x) = d_0 + d_1x + \cdots + d_{n+m}x^{m+n}$$

with $d_{n+m} = b_mc_n$.

Since $b_m \neq 0$ and $c_n \neq 0$ then $d_{n+m} \neq 0$.

So $b(x)c(x) \neq 0$.

So $\mathbb{F}[x]$ satisfies the cancellation law.

So $\mathbb{F}[x]$ is a PID //.

<u>Proposition</u> Let A be a commutative ring. Then A satisfies the cancellation law if and only if A has no zero divisors.

<u>Proof</u> $\Rightarrow$ Assume A satisfies the cancellation law.

To show: A has no zero divisors

To show: If $b, c \in A$ and $c \neq 0$ and $bc = 0$ then $b = 0$.

Assume $b, c \in A$ and $c \neq 0$ and $bc = 0$.

So

$$bc = 0 = 0 \cdot c.$$

Thus, by the cancellation law $b = 0$.

$\Leftarrow$ Assume A has no zero divisors.

To show: A satisfies the cancellation law.

To show: If $a, b, c \in A$ and $c \neq 0$ and $ac = bc$
then $a = b$.

Assume $a, b, c \in A$ and $c \neq 0$ and $ac = bc$.

Then $0 = bc - ac = (b-a)c$.

Since A has no zero divisors then $b - a = 0$.

So $b = a$.

So A satisfies the cancellation law. //

Partial fractions

Backwards of common denominator:

$$\frac{5x+22}{(x+2)(x+6)} = \frac{3}{x+2} + \frac{2}{x+6} \quad \text{or} \quad \frac{31}{33} = \frac{3}{11} + \frac{2}{3}$$

Splitting Let A be a PID and let $p,q \in A$ with $pA + qA = A$ (i.e. $\gcd(p,q) = 1$). Then let $r,s \in A$ such that

$$1 = pr + qs. \text{ Then } \frac{1}{pq} = \frac{r}{q} + \frac{s}{p}$$

and $$\frac{a}{pq} = \frac{ar}{q} + \frac{as}{p}.$$

Prime powers

$$\frac{a_1}{p} + \frac{a_2}{p^2} + \frac{a_3}{p^3} = \frac{a_1 p^2 + a_2 p + a_3}{p^3} \quad \text{with } a_1, a_2, a_3 \in A/pA$$

Representatives of A/qA

If $a = bq + r$ then $\frac{a}{q} = b + \frac{r}{q}$.

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Partial Fractions
A. Ram

Partial fractions for $\dfrac{2x^4+3x^2}{(x^2+1)^2(x^2+2)}$

Note that (say, by the Euclidean algorithm computation of the $\gcd((x^2+1)^2, x^2+2)$),

$$1 = (-x^2)(x^2+2) + (x^2+1)^2$$

so

$$\frac{2x^4+3x^2}{(x^2+1)^2(x^2+2)} = \frac{(2x^2-1)(x^2+2)+2}{(x^2+1)^2(x^2+2)} = \frac{2x^2-1}{(x^2+1)^2} + \frac{2}{(x^2+1)^2(x^2+2)}$$

$$= \frac{2(x^2+1)-3}{(x^2+1)^2} + \frac{2\left((-x^2)(x^2+2)+(x^2+1)^2\right)}{(x^2+1)^2(x^2+2)}$$

$$= \frac{2(x^2+1)-3-2x^2}{(x^2+1)^2} + \frac{2}{x^2+2} = \frac{-1}{(x^2+1)^2} + \frac{2}{x^2+2}$$