A <u>$n\times n$ matrix with entries in $\mathbb{F}$</u> is a function

$$X:\{1,\dots,n\}\times\{1,\dots,n\}\to\mathbb{F}$$

$$\begin{pmatrix} X(1,1) & \cdots & X(1,n) \\ \vdots & & \vdots \\ X(n,1) & & X(n,n) \end{pmatrix}$$

Let $V=M_n(\mathbb{F})$. Then V is an $\mathbb{F}$-vector space with

$$\begin{aligned} V\times V &\to V \\ (X,Y) &\mapsto X+Y \end{aligned} \quad \text{and} \quad \begin{aligned} \mathbb{F}\times V &\to V \\ (c,X) &\mapsto cX \end{aligned}$$

where

$$(X+Y)(i,j) = X(i,j)+Y(i,j) \quad \text{and}$$
$$(cX)(i,j) = cX(i,j)$$

Then $\dim V=n^2$. Define

$$\begin{aligned} \mathrm{tr}: V &\to \mathbb{C} \\ X &\mapsto \sum_{i=1}^{n} X(i,i) \end{aligned} \quad \text{so that}$$

$$\mathrm{tr}\,X = X(1,1)+X(2,2)+\cdots+X(n,n).$$

<u>Proposition</u> (a) tr is a linear transformation

(b) If $X,Y\in V$ then $\mathrm{tr}(XY)=\mathrm{tr}(YX)$.

Proof (a) To show: (aa) If $X, Y \in V$ then $tr(X+Y) = tr(X) + tr(Y)$.

(ab) If $c \in \mathbb{F}$ and $X \in V$ then $tr(cX) = c\,tr(X)$.

(aa) Assume $X, Y \in V$.

To show: $tr(X+Y) = tr(X) + tr(Y)$.

$$tr(X+Y) = \sum_{i=1}^{n} (X+Y)(i,i)$$
$$= \sum_{i=1}^{n} X(i,i) + Y(i,i)$$
$$= \sum_{i=1}^{n} X(i,i) + \sum_{i=1}^{n} Y(i,i)$$
$$= tr(X) + tr(Y).$$

(ab) Assuming $c \in \mathbb{F}$ and $X \in V$.

To show: $tr(cX) = c\,tr(X)$.

$$tr(cX) = \sum_{i=1}^{n} (cX)(i,i) = \sum_{i=1}^{n} cX(i,i)$$
$$= c\sum_{i=1}^{n} X(i,i) = c\,tr(X).$$

So $tr: V \to \mathbb{C}$ is a linear transformation.

(b) To show: If $X, Y \in V$ then $tr(XY) = tr(YX)$.

Assume $X, Y \in V$, ~~Then~~

To show: $tr(XY) = tr(YX)$.

$$tr(XY) = \sum_{i=1}^{n} (XY)(i,i) = \sum_{i=1}^{n}\Big(\sum_{j=1}^{n} X(i,j)Y(j,i)\Big)$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} Y(j,i)X(i,j)$$

$$= \sum_{j=1}^{n} (YX)(j,j) = tr(YX).$$

//.

Let $n=4$. Then

$$B = \left\{ \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix} \right\}$$ is a basis of V.

$C = \{6\}$ is a basis of $\mathbb{C}$.

The matrix of tr with respect to $\mathcal{B}$ and C is

$$tr_{C\mathcal{B}} = \left(\tfrac{1}{6}, 0, 0, \tfrac{1}{6}\right).$$

since

$$tr\begin{pmatrix}1&0\\0&0\end{pmatrix} = 1 = \tfrac{1}{6}\cdot 6, \quad tr\begin{pmatrix}0&1\\0&0\end{pmatrix} = 0$$

$$tr\begin{pmatrix}0&0\\1&0\end{pmatrix} = 0, \quad tr\begin{pmatrix}0&0\\0&1\end{pmatrix} = 1 = \tfrac{1}{6}\cdot 6.$$

Define $\langle\, , \rangle : V \times V \to \mathbb{F}$ by

$$\langle X, Y \rangle = tr(XY).$$

<u>Proposition</u> Let $X, Y, Z \in V$ and $c \in \mathbb{F}$. Then

(a) $\langle cX, Y \rangle = c\langle X, Y \rangle$.

(b) $\langle Y, X \rangle = \langle X, Y \rangle$

(c) $\langle X, cY \rangle = c\langle X, Y \rangle$

(d) $\langle X+Y, Z \rangle = \langle X, Z \rangle + \langle Y, Z \rangle$

(e) $\langle X, Y+Z \rangle = \langle X, Y \rangle + \langle X, Z \rangle$.

(f) If $X \in V$ and $X \neq 0$ then there exists $Y \in V$ such that $\langle X, Y \rangle \neq 0$.

(g) $V^{\perp} = \{0\}$ and $V \cap V^{\perp} = \{0\}$.

(h) If $n = 2$ then $\left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle = 0$.

<u>Proof</u> Assume $X, Y, Z \in V$ and $c \in \mathbb{F}$.

(a) $\langle cX, Y \rangle = tr(cXY) = c\, tr(XY) = c\langle X, Y \rangle$.

(b) $\langle Y, X \rangle = tr(YX) = tr(XY) = \langle X, Y \rangle$.

(c) $\langle X, cY \rangle = \langle cY, X \rangle = c\langle Y, X \rangle = c\langle X, Y \rangle$

(d) $\langle X+Y, Z \rangle = tr((X+Y)Z) = tr(XZ + YZ)$

$= tr(XZ) + tr(YZ) = \langle X, Z \rangle + \langle Y, Z \rangle$.

(e) $\langle X, Y+Z\rangle = \langle Y+Z, X\rangle = \langle Y, X\rangle + \langle Z, X\rangle$
$= \langle X, Y\rangle + \langle X, Z\rangle$.

(f) Let E_{kl} denote the matrix with 1 in the kl entry and 0 elsewhere.

To show: If $X \in V$ and $X \neq 0$ then there exists $Y \in V$ such that $\langle X, Y\rangle \neq 0$.

Assume $X \in V$ and $X \neq 0$.

To show: There exists $Y \in V$ such that $\langle X, Y\rangle \neq 0$.

Since $X \neq 0$ there exist $i, j \in \{1, \ldots, n\}$ such that $X(i,j) \neq 0$.

Let $Y = E_{ji}$.

To show: $\langle X, Y\rangle \neq 0$.

$$\begin{aligned}\langle X, Y\rangle &= tr(XY) = tr(XE_{ji}) = tr(XE_{j1}E_{1i}) \\ &= tr(E_{1i}XE_{j1}) = tr(E_{1i}E_{ii}XE_{jj}E_{j1}) \\ &= tr(E_{1i}X(i,j)E_{ij}E_{j1}) = tr(X(i,j)E_{1i}E_{ij}E_{j1}) \\ &= tr(X(i,j)E_{11}) = X(i,j)tr(E_{11}) = X(i,j) \neq 0.\end{aligned}$$

So $\langle X, Y\rangle \neq 0$.

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GTLA Lect. (6)

(ga) To show: $V^\perp = \{0\}$.

To show: If $X \in V^\perp$ then $X = 0$.

To show: If $X \neq 0$ then $X \notin V^\perp$.

Assume $X \neq 0$.

By (f) there exists $Y \in V$ such that $\langle X, Y \rangle \neq 0$.

So $X \notin V^\perp$.

So $V^\perp = \{0\}$.

(gb) Since $V^\perp = \{0\}$ then $V \cap V^\perp = \{0\}$.

(h) Assume $n=2$.

To show: $\left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle = 0$.

$$\left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle = tr\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = tr \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$