Let E_{ij} be the matrix with 1 in (i,j) entry and 0 elsewhere.

Then

$$\langle E_{ij}, E_{kl} \rangle = tr(E_{ij} E_{kl}) = \begin{cases} 0, & \text{if } j \neq k \text{ or } i \neq l, \\ 1, & \text{if } j = k \text{ and } i = l. \end{cases}$$

So the dual basis to $\{E_{ij} \mid i,j \in \{1,2,\dots,n\}\}$ is

$$\{E_{ji} \mid j,i \in \{1,2,\dots,n\}\}.$$

Let $A \in M_n(\mathbb{F})$. Define

$$f: V \to V \text{ by}$$
$$X \mapsto AX - XA$$

Recall that $\langle , \rangle : V \times V \to \mathbb{F}$ given by

$$\langle X, Y \rangle = tr(XY)$$

Since $\langle f(X), Y \rangle = \langle AX - XA, Y \rangle = tr((AX - XA)Y)$

$$= tr(AXY - XAY) = tr(XYA - XAY)$$

$$= \langle X, -f(Y) \rangle$$

So $f^* = -f$.

Let $n=2$ and $A=\begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix}$.

$V = \text{span}\{E_{11}, E_{12}, E_{21}, E_{22}\}$.

$= \text{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\} = M_2(\mathbb{F})$.

The matrix of f with respect to

$$B = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

is

$$f_{BB} = \begin{pmatrix} 0 & 3 & 12 & 0 \\ -12 & 12 & 0 & 12 \\ -3 & 0 & -12 & 3 \\ 0 & -3 & -12 & 0 \end{pmatrix}$$

since

$$f(E_{11}) = \begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ -3 & 0 \end{pmatrix} - \begin{pmatrix} 6 & 12 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -12 \\ -3 & 0 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - 12\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - 3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f(E_{12}) = \begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ 0 & -3 \end{pmatrix} - \begin{pmatrix} -3 & -6 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 12 \\ 0 & -3 \end{pmatrix} = 3\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 12\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + -3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f(E_{21}) = \begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix} = \begin{pmatrix} 12 & 0 \\ -6 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 6 & 12 \end{pmatrix}$$

30.10.2020 (3)
GTLALect

$$= \begin{pmatrix} 12 & 0 \\ -12 & -12 \end{pmatrix} = 12\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - 12\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - 12\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f(E_{22}) = \begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 12 \\ 0 & -6 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -3 & -6 \end{pmatrix} = \begin{pmatrix} 0 & 12 \\ 3 & 0 \end{pmatrix}$$

$$= 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 12\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Elementary matrices</u> $x_{ij}(c)$

$$\begin{pmatrix} | & | & & | \\ p_1 & p_2 & \cdots & p_n \\ | & | & & | \end{pmatrix}\begin{pmatrix} 1 & & c & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} | & & | & & | \\ p_1 & \cdots & cp_i + p_j & \cdots & p_n \\ | & & | & & | \end{pmatrix}$$