21.09.2023
GTLA ①
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Proposition Let G be a group and let N be a subgroup of G. Then

N is a normal subgroup of G

if and only if

$$\begin{matrix} G/N \times G/N & \to & G/N \\ (aN, bN) & \mapsto & abN \end{matrix}$$ is a function.

Proof $\Rightarrow$: Assume N is a normal subgroup of G.

To show: $$\begin{matrix} G/N \times G/N & \to & G/N \\ (aN, bN) & \mapsto & abN \end{matrix}$$ is a function.

To show: If $(a_1N, b_1N), (a_2N, b_2N) \in G/N \times G/N$ and $(a_1N, b_1N) = (a_2N, b_2N)$ then $a_1b_1N = a_2b_2N$

Assume $(a_1N, b_1N), (a_2N, b_2N) \in G/N \times G/N$ and $(a_1N, b_1N) = (a_2N, b_2N)$.

To show: $a_1b_1N = a_2b_2N$.

Since $(a_1N, b_1N) = (a_2N, b_2N)$ then

$a_1N = a_2N$ and $b_1N = b_2N$.

[illegible] there exist $n_1, n_2 \in N$ such that $a_1 = a_1 \cdot 1 = a_2n_1$ and $b_1 = b_1 \cdot 1 = b_2n_2$.

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To show: (a) $a_1 b_1 N \subseteq a_2 b_2 N$

(b) $a_2 b_2 N \subseteq a_1 b_1 N$

(a) Let $k \in a_1 b_1 N$

Then there exists $n \in N$ such that $k = a_1 b_1 n$.

To show: $k \in a_2 b_2 N$.

$k = a_1 b_1 n = a_2 n_1 b_2 n_2 n$

$= a_2 b_2 b_2^{-1} n_1 b_2 n_2 n$

$= a_2 b_2 n_3$, where

$n_3 = (b_2^{-1} n_1 b_2) n_2 n \in N$, since N is normal.

So $a_1 b_1 N \subseteq a_2 b_2 N$

(b) Let $k \in a_2 b_2 N$

Then there exists $n \in N$ such that $k = a_2 b_2 n$

To show: $k \in a_1 b_1 N$.

$k = a_2 b_2 n = a_1 n_1^{-1} b_1 n_2^{-1} n$

$= a_1 b_1 b_1^{-1} n_1^{-1} b_1 n_2^{-1} n$

$= a_1 b_1 n_3$, where

$n_3 = (b_1^{-1} n_1^{-1} b_1) n_2^{-1} n$, since N is a normal subgroup.

So $a_2 b_2 N \subseteq a_1 b_1 N$.

So $a_1 b_1 N = a_2 b_2 N$.

So $\begin{matrix} G/N \times G/N & \to & G/N \\ (aN, bN) & \mapsto & abN \end{matrix}$ is a function.

$\Leftarrow$ To show: If $\begin{matrix} G/N \times G/N & \to & G/N \\ (aN, bN) & \mapsto & abN \end{matrix}$ is a function

then N is a normal subgroup of G.

Assume $\begin{matrix} G/N \times G/N & \to & G/N \\ (aN, bN) & \mapsto & abN \end{matrix}$ is a function.

To show: N is a normal subgroup of G

To show: If $g \in G$ and $n \in N$ then $gng^{-1} \in N$

Assume $g \in G$ and $n \in N$.

To show: $gng^{-1} \in N$.

We know

$$(gN\, nN)\, g^{-1}N = gnN \cdot g^{-1}N = gng^{-1}N$$

and

$$gN\, nN\, g^{-1}N = gN \cdot 1N \cdot g^{-1}N = gN\, g^{-1}N = N.$$

Note: If $k \in nN$ then $k = nn' \in N$ so that $nN \subseteq N$
and if $k \in N$ then $k = nn^{-1}k \in nN$ so that $N \subseteq nN$.

So $gng^{-1} \in gng^{-1}N = N$.

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So N is a normal subgroup of G. //. A. Rain

Let
$$G = \{\equiv, \underline{\times}, \overline{\times}, \overline{\underline{\times}}, \ast, [illegible], \underline{\ast}\}$$
$$N = \{\equiv, \underline{\times}\}. \qquad \text{Let } C_g = gN = \{gn \mid n \in N\}.$$
Then
$$G/N = \{C_{\equiv}, C_{\overline{\times}}, C_{\ast}\} \text{ where}$$
$$C_{\equiv} = \{\equiv, \underline{\times}\}, \quad C_{\overline{\times}} = \{\overline{\times}, \overline{\underline{\times}}\}, \quad C_{\ast} = \{\ast, \underline{\ast}\}.$$
If
$$C_{\underline{\times}} = \{\underline{\times}, \equiv\}, \quad C_{\overline{\underline{\times}}} = \{\overline{\underline{\times}}, \overline{\times}\}, \quad C_{\underline{\ast}} = \{\underline{\ast}, \ast\}$$
then
$$C_{\underline{\times}} = C_{\equiv}, \quad C_{\overline{\underline{\times}}} = C_{\overline{\times}}, \quad C_{\ast} = C_{\underline{\ast}}.$$
Define
$$G/N \times G/N \longrightarrow G/N$$
$$(C_a, C_b) \longmapsto C_{ab}.$$
Then
$$C_{\underline{\times}} C_{\underline{\times}} = C_{\equiv} \text{ and } C_{\overline{\times}} C_{\overline{\times}} = C_{\equiv}$$
$$\text{and } C_{\overline{\underline{\times}}} C_{\overline{\underline{\times}}} = C_{\underline{\ast}} \neq C_{\equiv} \quad \text{since } \overline{\underline{\times}}\,\overline{\underline{\times}} = \underline{\ast}$$
So
$$G/N \times G/N \to G/N$$
$$(C_a, C_b) \mapsto C_{ab} \text{ is not a function.}$$

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Let S and T be sets.

A function from S to T $f: S \to T$ is a subset $\Gamma \subseteq S \times T$ such that

(a) If $s \in S$ then there exists $t \in T$ such that $(s,t) \in \Gamma$.

(b) If $(s,t_1) \in \Gamma$ and $(s,t_2) \in \Gamma$ then $t_1 = t_2$.

The conversion between $\begin{array}{c} f: S \to T \\ s \mapsto f(s) \end{array}$ and Γ is

$$\Gamma = \{(s, f(s)) \mid s \in S\}$$

and

$$f(s) = t \quad \text{if } (s,t) \in \Gamma.$$

A subgroup N is normal if N satisfies:

if $g \in G$ and $n \in N$ then $gng^{-1} \in N$.

A subgroup of G is a subset $N \subseteq G$ such that

(a) If $n_1, n_2 \in N$ then $n_1 n_2 \in N$.

(b) If $n \in N$ then $n^{-1} \in N$.

Multiplication table for S_3. 21.09.2023 GTLA A. Ram

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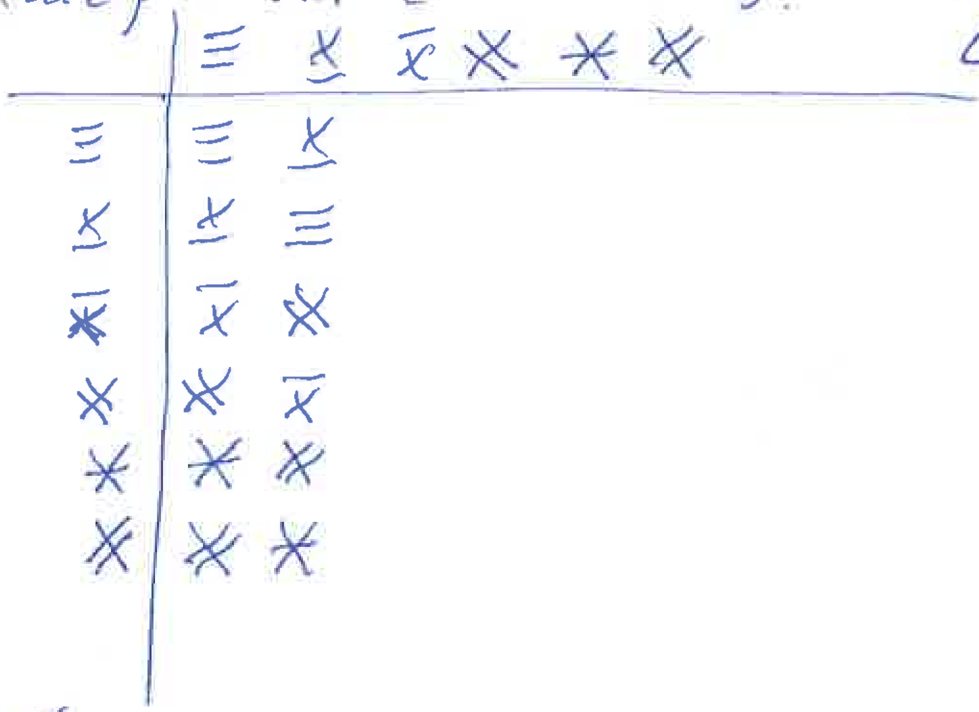

For example