MAST20022 Group Theory and Linear Algebra

Assignment 2

Due: 4pm Wednesday 16 September 2020

1. Consider the matrix $M = \begin{bmatrix} i & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & i \end{bmatrix} \in M_3(\mathbb{C})$.
   (a) Find the minimal polynomial of M.
   (b) Use your answer for part (a) to determine whether M is diagonalizable.
   (c) Find the Jordan normal form of M.

2. Let $A \in M_n(\mathbb{C})$ and suppose that $A^3 = A^2$. Show that A^2 is diagonalisable and $A^2 - A$ is nilpotent.

3. Let $A \in M_3(\mathbb{R})$ be given by
$$A = \begin{bmatrix} 5 & 5 & -3 \\ -2 & -3 & 2 \\ 4 & 2 & -1 \end{bmatrix}$$
   (a) Show that $X - 1$ and $X^2 + 1$ are relatively prime in $\mathbb{R}[X]$.
   (b) Given that the characteristic polynomial of A is $(X-1)(X^2+1)$, find the minimal polynomial $m(X) \in \mathbb{R}[X]$ of A.
   (c) Find matrices $B \in M_1(\mathbb{R})$, $C \in M_2(\mathbb{R})$, and $P \in M_3(\mathbb{R})$ such that P is invertible and
$$P^{-1}AP = B \oplus C$$
   .

4. Suppose $f \colon \mathbb{C}^5 \to \mathbb{C}^5$ is a linear transformation such that $f^5 = f^4$ and $\dim(\ker(f)) = 3$.
   (a) Describe all the possibilities for the Jordan normal form of f that are compatible with these conditions.
   (b) Suppose that, in addition to the above conditions, f satisfies $\dim(\ker(f^2)) = 5$. What now are the possibilities for the Jordan normal form of f?

5. Let
$$\mathbb{A} = \{c + d\sqrt{-5} \mid c, d \in \mathbb{Z}\} \qquad \text{and} \qquad \mathbb{F} = \{c + d\sqrt{-5} \mid c, d \in \mathbb{Q}\}.$$
   (a) Let $c, d \in \mathbb{Z}$, not both 0. Find the inverse of $c + d\sqrt{-5}$ in $\mathbb{F}$.
   (b) Show that the only invertible elements of $\mathbb{A}$ are 1 and -1.
   (c) Show that 2 cannot be factored in $\mathbb{A}$ (without using an invertible element).
   (d) Check that, in $\mathbb{A}$, $6 = (1+\sqrt{-5})(1-\sqrt{-5})$ and $6 = 2 \cdot 3$ and then show that 2 is not a factor of $1 + \sqrt{-5}$ and 2 is not a factor of $1 - \sqrt{-5}$.