Problem sheet 10

$\mathbb{E}^2$ and isometries

Vocabulary

(1) Define $\mathbb{R}^2$ and $\mathbb{E}^2$ and give some illustrative examples.

(2) Define isometry of $\mathbb{E}^2$ and give some illustrative examples.

(3) Define a rotation of $\mathbb{E}^2$ and give some illustrative examples.

(4) Define a reflection of $\mathbb{E}^2$ and give some illustrative examples.

(5) Define a translation of $\mathbb{E}^2$ and give some illustrative examples.

(6) Define glide reflection of $\mathbb{E}^2$ and give some illustrative examples.

(7) Define $\mathbb{R}^n$ and $\mathbb{E}^n$ and give some illustrative examples.

(8) Define isometry of $\mathbb{E}^n$ and give some illustrative examples.

(9) Define a rotation of $\mathbb{E}^n$ and give some illustrative examples.

(10) Define a reflection of $\mathbb{E}^n$ and give some illustrative examples.

(11) Define a translation of $\mathbb{E}^n$ and give some illustrative examples.

(12) Define the groups $O_n(\mathbb{R})$ and $SO_n(\mathbb{R})$ and give some illustrative examples.

(13) Define a rotation in $\mathbb{R}^2$ and give some illustrative examples.

(14) Define a rotation in $\mathbb{R}^3$ and give some illustrative examples.

Results

(1) Show that if an isometry fixes two points then it fixes all points of the line on which they lie.

(2) Show that if an isometry fixes three points which do not all lie on a line then it fixes all of $\mathbb{E}^2$.

(3) Let σ_1 and σ_2 be reflections in axes L_1 and L_2. Show that
   (a) If L_1 and L_2 intersect then the product $\sigma_1\sigma_2$ is a rotation about the point of intersection of L_1 and L_2 with an angle of rotation twice the angle between L_1 and L_2, and

(b) If L_1 and L_2 are parallel then the product $\sigma_1\sigma_2$ is a translation in a direction perpendicular to L_i with a magnitude equal to twice the distance between L_1 and L_2.

(4) Show that the product of three reflections in parallel axes is a reflection.

(5) Show that the product of three reflections in axes which are not parallel and which do not intersect in a point is a glide reflection.

(6) Show that the set of fixed points of an isometry is one of the following:
   (a) All of $\mathbb{E}^2$, in which case the isometry is the identity;
   (b) A line in $\mathbb{E}^2$, in which case the isometry is the reflection in that line;
   (c) A single point, in which case the isometry is a rotation about that point and can be expressed as the product of two reflections;
   (d) empty, in which case the isometry is either
      (A) a translation and can be expressed as the product of two reflections or
      (B) a glide reflection and can be expressed as the product of three reflections.

(7) Let $\mathcal{I}$ be the group of isometries of $\mathbb{E}^2$. Show that the set of translations forms a normal subgroup of $\mathcal{I}$.

(8) Let $\mathcal{I}$ be the group of isometries of $\mathbb{E}^2$. Let P be a point of $\mathbb{E}^2$. Show that the set of isometries of $\mathbb{E}^2$ which fix P is a subgroup of $\mathcal{I}$.

(9) Let $\mathcal{I}$ be the group of isometries of $\mathbb{E}^2$. Let P and Q be points of $\mathbb{E}^2$. Let $\mathcal{O}_P$ be the set of isometries that fix P and let $\mathcal{O}_Q$ be the set of isometries that fix Q. Show that $\mathcal{O}_P$ and $\mathcal{O}_Q$ are conjugate subgroups of ?.

(10) Let $\mathcal{I}$ be the group of isometries of $\mathbb{E}^2$. Let P be a point of $\mathbb{E}^2$. Show that every element of $\mathcal{I}$ can be uniquely expressed as a product of a translation and an isometry fixing P.

(11) Let $\mathcal{I}$ be the group of isometries of $\mathbb{E}^2$. Let P be a point of $\mathbb{E}^2$. Let $\mathcal{O}_P$ be the set of isometries that fix P. Show that there is a surjective homomorphism $\pi_P \colon \mathcal{I} \to \mathcal{O}_P$.

(12) Show that a finite group of isometries of $\mathbb{E}^2$ is a cyclic group or a dihedral group.

(13) Let f be an isometry of $\mathbb{E}^n$ such that $f(0) = 0$. Show that there exists an orthogonal matrix $A \in O_n(\mathbb{R})$ such that $f(x) = Ax$, for $x \in \mathbb{E}^n$.

(14) Show that if $f \colon \mathbb{E}^n \to \mathbb{E}^n$ then there exist $A \in O_n(\mathbb{R})$ and $b \in \mathbb{R}^n$ such that $f(x) = Ax + b$.

Examples and computations

(1) Describe the rotational symmetries of a cube. There are 24 in all. Are there any other symmetries besides these rotations?

(2) Describe the 12 rotational symmetries of a regular tetrahedron.

(3) Find two ?different? multiplication tables for groups with 4 elements. Show that both can be represented as symmetry groups of geometric figures in $\mathbb{R}^2$.

(4) Let $A \in O_n(\mathbb{R})$. Show that the linear transformation

$$f\colon \mathbb{R}^n \to \mathbb{R}^n \qquad \text{defined by} \quad f(x) = Ax$$

is an isometry.

(5) Let $b \in \mathbb{R}^n$. Show that the function

$$t_b\colon \mathbb{R}^n \to \mathbb{R}^n \qquad \text{given by} \qquad t_b(x) = x + b$$

is an isometry. Show that the inverse of t_b is t_{-b}.

(6) Show that compositions of isometries are isometries.

(7) Define a "reflection in a line" in $\mathbb{E}^2$ and show that it is an isometry.

(8) Define a "rotation about a point" in $\mathbb{E}^2$ and show that it is an isometry.

(9) Define a "translation" in $\mathbb{E}^2$ and show that it is an isometry.

(10) Define a "glide relfection" in $\mathbb{E}^2$ and show that it is an isometry.

(11) Let $\mathcal{I}$ be the group of isometries of $\mathbb{E}^2$. Let $\mathcal{I}_+$ denote the subset of $\mathcal{I}$ consisting of all translations together with all rotations. Show that $\mathcal{I}_+$ is a subgroup of $\mathcal{I}$.

(12) Let $\mathcal{I}$ be the group of isometries of $\mathbb{E}^2$. Let $\mathcal{I}_+$ denote the subset of $\mathcal{I}$ consisting of all translations together with all rotations. Show that $\mathcal{I}_+$ is a subgroup of index 2 in $\mathcal{I}$ and that $\mathcal{I}_+$ is a normal subgroup of $\mathcal{I}$.

(13) Let $\mathcal{I}$ be the group of isometries of $\mathbb{E}^2$. Let $\mathcal{I}_+$ denote the subset of $\mathcal{I}$ consisting of all translations together with all rotations. Show that $f \in \mathcal{I}_+$ if and only if f is a product of an even number of reflections.

(14) Identify $\mathbb{E}^2$ with the complex plane so that each point of $\mathbb{E}^2$ can be represented by a complex number. Show that every isometry can be represented in the form $z \mapsto e^{i\theta}z + u$ or of the form $z \mapsto e^{i\theta}\bar{z} + u$, for some real number θ and some complex number u. Show that the former type correspond to orientation preserving isometries.

(15) Let $\mathcal{I}$ be the group of isometries of $\mathbb{E}^2$. Describe the conjugacy classes in the group $\mathcal{I}$.

(16) Show that if f and g are isometries of $\mathbb{E}^n$ then so is $f \circ g$.

(17) Let (A, b) denote the isometry of $\mathbb{E}^n$ given by $x \mapsto Ax + b$ for $A \in O(n)$, $b \in \mathbb{R}^n$.
   (a) Show that the function $\pi\colon \text{isom}(\mathbb{E}^n) \mapsto O(n)$ given by $\pi((A, b)) = A$ is a homomorphism.
   (b) Find the kernel and image of π.

(c) Deduce that the set T of all translations is a normal subgroup of $\mathrm{isom}(\mathbb{E}^n)$ with $\mathrm{isom}(\mathbb{E}^n)/T$ isomorphic to $O(n)$.

(18) Show that the subset $\mathrm{isom}_+(\mathbb{E}^n)$ of orientation preserving isometries of $\mathbb{E}^n$ is a normal subgroup of index 2 in $\mathrm{isom}(\mathbb{E}^n)$.

(19) Write each of the following isometries of $\mathbb{E}^2$ in the form (A, b), where $A \in O(2)$ and $b \in \mathbb{R}^2$.

(i) f is the anticlockwise rotation through $\pi/2$ about the point $(0, 0)$.

(ii) g is the anticlockwise rotation through π about the point $(1, 0)$.

(iii) h is the reflection in the line $x + y + 2 = 0$.

(iv) f, g and $g \circ f$.

(20) Let f and g be the isometries of $\mathbb{E}^2$ given by: f is the anticlockwise rotation through $\pi/2$ about the point $(0, 0)$ and g is the anticlockwise rotation through π about the point $(1, 0)$. Show that $f \circ g$ and $g \circ f$ are rotations and find the fixed point and the angle of rotation for each of them.

(21) Let R_1 and R_2 be reflections in the lines $y = 0$ and $y = a$, respectively. Find formulas for R_1 and R_2 and verify that $R_1 \circ R_2$ and $R_2 \circ R_1$ are translations.

(22) Let f be an orientation reversing isometry of $\mathbb{E}^2$. Show that f^2 is a translation.

(23) Let $\mathrm{Fix}(h) = \{x \mid h(x) = x\}$. Show that if $f : \mathbb{E}^2 \to \mathbb{E}^2$ and $g : \mathbb{E}^2 \to \mathbb{E}^2$ are isometries then $\mathrm{Fix}(gfg^{-1}) = gFix(f)$.

(24) Let $f : \mathbb{E}^2 \to \mathbb{E}^2$ and $g : \mathbb{E}^2 \to \mathbb{E}^2$ be isometries. Show that if f is the reflection in a line L then gfg^{-1} is reflection in the line $g(L)$.

(25) Let $f : \mathbb{E}^2 \to \mathbb{E}^2$ and $g : \mathbb{E}^2 \to \mathbb{E}^2$ be isometries. Show that if f is a rotation by θ about p then gfg^{-1} is a rotation about $g(p)$ by θ if g preserves orientation and by $-\theta$ if g reverses orientation.

(26) Let $f : \mathbb{E}^2 \to \mathbb{E}^2$ and $g : \mathbb{E}^2 \to \mathbb{E}^2$ be isometries. Show that if f is a translation then gfg^{-1} is a translation by the same distance.

(27) Let D_∞ be the set of isometries of $\mathbb{E}^2$ consisting of all translations by $(n, 0)$ and all reflections in the lines $x = n/2$, where $n \in \mathbb{Z}$. Show that D_∞ is a subgroup of $\mathrm{isom}(\mathbb{E}^2)$.

(28) Let D_∞ be the set of isometries of $\mathbb{E}^2$ consisting of all translations by $(n, 0)$ and all reflections in the lines $x = n/2$, where $n \in \mathbb{Z}$. Show that D_∞ acts on the x-axis and find the orbit and stabilizer of each of the points $(1, 0)$, $(\frac{1}{2}, 0)$, $(\frac{1}{3}, 0)$.

(29) Let D_∞ be the set of isometries of $\mathbb{E}^2$ consisting of all translations by $(n, 0)$ and all reflections in the lines $x = n/2$, where $n \in \mathbb{Z}$. Show that D_∞ is generated by $a : (x, y) \mapsto (x + 1, y)$ and $b : (x, y) \mapsto (-x, y)$ and that these satisfy the relations $b^2 = 1$ and $bab^{-1} = a^{-1}$.

(30) Show that every orientation preserving isometry of $\mathbb{E}^3$ is either:

(i) a rotation about an axis,

(ii) a translation, or

(iii) a screw motion consisting of a rotation about an axis composed with a translation parallel to that axis.

(31) Show that a rotation fixing the origin on $\mathbb{R}^3$ has an eigenvalue 1. Show that the corresponding eigenspace is of dimension 1, the axis of rotation.

(32) Show that a rotation fixing the origin on $\mathbb{R}^2$ has two eigenvalues 1 and -1. Show that the eigenspace corresponding to 1 is the line of reflection and that the eigenspace corresponding to -1 is the perpendicular to the line of reflection.

(33) Let f be a rotation on $\mathbb{R}^3$. Then the plane perpendicular to the axis of rotation is an invariant subspace of f. Show that the matrix for the rotation with respect to a basis of two orthonormal vectors from the plane and a unit vector along the axis of rotation is

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(34) Let f be a reflection, in a line through the origin, in $\mathbb{R}^2$. Show that the minimal polynomial of f is $x^2 - 1$.

(35) Define a 4-dimensional cube and work out some of its rotational symmetries.

(36) What letters in the Roman alphabet display symmetry?

(37) Show that the set of all rotations of the plane about a fixed center P, together with the operation of composition of symmetries, form a group. What about all of the reflections for which the axis (or mirror) passes through P?

(38) Describe the product of a rotation of the plane with a translation. Describe the product of two (planar) rotations about different axes.

(39) Find the order of a reflection.

(40) Find the order of a translation in the group of symmetries of a plane pattern.

(41) Can you find an example of two symmetries of finite order where the product is of infinite order?

(42) Let G be the group of symmetries of a plane tesselation. Decide whether the set of rotations in G is a subgroup.