Problem sheet 4

Eigenvectors, minimal and characteristic polynomials

Vocabulary

(1) Define eigenvalue, eigenvector and eigenspace and give some illustrative examples.

(2) Define generalised eigenspace and give some illustrative examples.

(3) Define f-invariant subspace and restriction of f and give some illustrative examples.

(4) Define complement (to a subspace) and give some illustrative examples.

(5) Define monic polynomial and give some illustrative examples.

(6) Define minimal polynomial and characteristic polynomial and give some illustrative examples.

(7) Define invertible matrix and give some illustrative examples.

(8) Define define diagonal matrix, upper triangular matrix, strictly upper triangular matrix, and unipotent upper triangular matrix and give some illustrative examples.

(9) Let $\mathbb{F}$ be a field and let $d, a \in \mathbb{F}[t]$. Define the ideal generated by d and "d divides a" and give some illustrative examples.

(10) Let $\mathbb{F}$ be a field and let $x, m \in \mathbb{F}[t]$. Define the greatest common divisor of x and m and give some illustrative examples. Let $\mathbb{F}$ be a field and let $p \in \mathbb{F}[t]$. Define the degree of p and monic polynomial and give some illustrative examples.

Results

(1) Let $\mathbb{F}$ be a field and let $a, b \in \mathbb{F}[t]$. Show that there exist $q, r \in \mathbb{F}[t]$ such that

(a) $a = bq + r$,

(b) $r = 0$ or $\deg(r) < \deg(b)$.

(2) Let $a, b \in \mathbb{Z}$. Show that there exist $q, r \in \mathbb{Z}$ such that

(a) $a = bq + r$, and

(b) $0 < r < |b|$.

(3) Let $a, b \in \mathbb{F}[t]$ and let $d = \gcd(a, b)$. Show that

(a) There exists a monic polynomial $\ell \in \mathbb{F}[t]$ such that $\ell\mathbb{F}[t] = a\mathbb{F}[t] + b\mathbb{F}[t]$, and

(b) $d = \ell$.

(4) Let $a, b \in \mathbb{Z}$ and let $d = \gcd(a, b)$. Show that
   (a) There exists a $\ell \in \mathbb{Z}$ such that $\ell\mathbb{Z} = a\mathbb{Z} + b\mathbb{Z}$, and
   (b) $d = \ell$.

(5) Let $f : V \to V$ be a linear transformation and let W be an f-invariant subspace with $\dim(V) = n$ and $\dim(W) = m$. Let $\mathcal{B}_1 = \{w_1, \ldots, w_m\}$ be a basis for W and extend it to a basis Let $\mathcal{B} = \{w_1, \ldots, w_m, w_{m+1}, \ldots, w_n\}$ for V. Show that the matrix of f with respect to $\mathcal{B}$ is of the block form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix},$$

where A, B, D are matrices and A is the $m \times n$ matrix of f_W with respect to the basis $\mathcal{B}_1$.

(6) Let V be a finite dimensional vector space and let U and W be subspaces of V. Show that the following are equivalent.
   (1) U is a complement of W.
   (2) There is a basis $\mathcal{B}$ of V of the form $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where $\mathcal{B}_1$ is a basis of U, $\mathcal{B}_2$ is a basis of W and $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$.
   (3) $U \cap W = \{0\}$ and $\dim(U) = \dim(V) - \dim(W)$,
   (4) $V = U + W$ and $\dim(U) = \dim(V) - \dim(W)$.

(7) Let V be a vector space and let f be a linear transformation on V. Let U and W be complementary subspaces of V. Suppose that both U and W are are f-invariant. Choose an ordered basis $\mathcal{B}$ of V of the form $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$ where $\mathcal{B}_1$ is a basis of U and $\mathcal{B}_2$ is a basis of W. Show that the matrix of f with respect to $\mathcal{B}$ is of the "block diagonal" form:

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

where A is the matrix of f_U and D is the matrix of f_W.

(8) Let $\mathbb{F}$ be a field and let V be a vector space over $\mathbb{F}$. Let $f : V \to V$ be a linear transformation and let $m(x)$ be the minimal polynomial of f. Show that if $q(x)$ is a polynomial with coefficients in $\mathbb{F}$ such that $q(f) = 0$ then $m(x)$ divides $q(x)$.

(9) Let $\mathbb{F}$ be a field and let V be a vector space over $\mathbb{F}$. Let $f : V \to V$ be a linear transformation and let $m(x)$ be the minimal polynomial of f. Show that the roots of $m(x)$ are exactly the eigenvalues of f.

(10) Let $\mathbb{F}$ be a field and let V be a vector space over $\mathbb{F}$ Let $f : V \to V$ be a linear transformation and let $p(x) \in \mathbb{F}[x]$. Show that the null space of $p(f)$ is an f-invariant subspace of V.

(11) Let $\mathbb{F}$ be a field and let V be a vector space over $\mathbb{F}$. Let $f : V \to V$ be a linear transformation and let $m(x)$ be the minimal polynomial of f. Suppose that $m(x)$ can be factored as $m(x) = p(x)q(x)$, where $p(x)$ and $q(x)$ are polynomials with coefficients in $\mathbb{F}$ which have no common factor. Show that V is a direct sum of f-invariant subspaces

$$V = W_p \oplus W_q,$$

where W_p and W_q are the nulspaces of $p(f)$ and $q(f)$, respectively. Show that the restrictions f_{W_p} and f_{W_q} have minimal polynomials $p(x)$ and $q(x)$, respectively.

(12) Let $\mathbb{F}$ be a field and let V be a vector space over $\mathbb{F}$. Let $f\colon V \to V$ be a linear transformation and let $m(x)$ be the minimal polynomial of f. Suppose that $m(x) = q_1(x)q_2(x)\cdots q_k(x)$, where $q_i(x)$ has no common factor with $q_j(x)$ if $i \neq j$. Let W_i be the nullspace of $q_i(f)$. Suppose that $\mathcal{B}_i$ is an ordered basis of W_i. Show that $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_k)$ is an ordered basis for V and the matrix of f with respect to $\mathcal{B}$ is

$$\begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_k \end{pmatrix},$$

where A_i is the matrix of f_{W_i} with respect to $\mathcal{B}_i$.

Examples and computations

(1) Let $b = t^3 - 10t^2 + 23t - 14$ and $a = t^4 - 3t^2 + 3t^2 - 3t + 2$. Find $d = \gcd(a, b)$ and find $x, y \in \mathbb{Q}[t]$ such that $d = ax + by$.

(2) Let $b = t^3 - 6t^2 + t + 4$ and $a = t^5 - 6t + 1$. Find $d = \gcd(a, b)$ and find $x, y \in \mathbb{Q}[t]$ such that $d = ax + by$.

(3) The eigenvalues of the (linear transformation corresponding to the) matrix

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

satisfy $\det(A - \lambda I) = 0$. Determine the eigenvalues and show that the corresponding eigenspaces are dimension 1 and are generated by the eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -7 \\ 8 \\ 2 \end{pmatrix},$$

(4) Let $C^\infty(\mathbb{R})$ be the space of functions $f\colon \mathbb{R} \to \mathbb{R}$ which are differentiable infinitely often. Show that the eigenvectors of differentiation are the functions e^{ax} for $a \in \mathbb{R}$. Determine the eigenvalues.

(5) Suppose that a linear transformation on $\mathbb{R}^3$ has matrix

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

with respect to the basis $\{e_1 = (1, 0, 0),\ e_2 = (0, 1, 0),\ e_3 = (0, 0, 1)\}$. Show that the subspace $W = \operatorname{span}\{e_1, e_2\}$ is f-invariant and that the matrix of f_W with respect to the basis $\{e_1, e_2\}$ is

$$\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}.$$

(6) Show that, in $\mathbb{R}^3$, a complement to a plane through the origin is any line through the origin which does not lie in the plane.

(7) Show that, in $\mathbb{R}^4$, the subspaces $\text{span}\{(1,0,0,0),(0,1,0,0)\}$ and $\text{span}\{(0,0,1,0),(0,0,0,1)\}$ are complementary.

(8) Show that, in $\mathbb{R}[x]$, the subspaces $\text{span}\{2, 1+x, 1+x+x^3\}$ and $\text{span}\{x^2+3x^4, x^4, x^5, x^6, \ldots\}$ are complementary.

(9) Let f be a linear transformation with matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Show that the minimal polynomial of f is $(x-2)(x-3)x$.

(10) Find the minimal polynomial of the matrix $\begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}$.

(11) Find the minimal polynomial of the matrix $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$.

(12) Find the minimal polynomial of the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

(13) Find the minimal polynomial of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$.

(14) Show that the matrices

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

have the same minimal polynomial and different characteristic polynomial.

(15) Show that the matrix

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

has minimal polynomial $x^2 - 2x - 8$. Use this to determine the inverse of A.

(16) Show that a linear transformation f is invertible if and only if its minimal polynomial has non-zero constant term. Assuming f is invertible, how can the inverse be calculated if the minimal polynomial is known?

(17) Suppose that A is an $n \times n$ upper triangular matrix with zeroes on the diagonal. Prove that $A^n = 0$.

(18) Let f be a linear transformation on a vector space V with minimal polynomial $x^2 - 1$. Suppose that $2 \neq 0$ in the field of scalars. (Thus, for example, $\mathbb{Z}/2\mathbb{Z}$ is not allowed as the

field of scalars.) Show directly that the subspaces

$$\{v \in V \mid f(v) = v\} \qquad \text{and} \qquad \{v \in V \mid f(v) = -v\}$$

are complementary subspaces of V. Find a diagonal matrix representing f.

(19) Let $\mathcal{P}_n(\mathbb{R})$ be the vector space of polynomials in $\mathbb{R}[x]$ of degree $\leqslant n$. Show that the linear transformation $\mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R})$ given by differentiation with respect to x cannot be represented by a diagonal matrix.

(20) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $T(x, y) = (x + 2y, -x, 0)$. Find the matrix of T with respect to the (ordered) bases $B = \{(1, 3), (-2, 4)\}$ for $\mathbb{R}^2$ and $C = \{(1, 1, 1), (2, 2, 0), (4, 0, 0)\}$ for $\mathbb{R}^3$.

(21) Let V be the subspace of functions from $\mathbb{R}$ to $\mathbb{R}$ spanned by $\{e^{2t}, te^{2t}, t^2e^{2t}\}$. Show that differentation with respect to t is well defined linear transformation D on V and find the matrix of D with respect to the basis $\{e^{2t}, te^{2t}, t^2e^{2t}\}$ of V.

(22) Find the minimal polynomial of the matrix $\begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$.

(23) Find the minimal polynomial of the matrix $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.

(24) Find the minimal polynomial of the matrix $\begin{pmatrix} 2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.

(25) Let W_1 and W_2 be subspaces of a vector space V. Show that V is the direct sum of W_1 and W_2 if and only if every vector $v \in V$ can be written uniquely in the form $v = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$.

(26) (i) Show that the complex numbers $\mathbb{C}$ is a vector space over the field of real numbers $\mathbb{R}$.

(ii) Show that $\{1, i\}$ is a basis for $\mathbb{C}$ over $\mathbb{R}$.

(iii) Let $\alpha = a + ib$ be a complex number. Show that multiplication by α is a linear transformation $f\colon \mathbb{C} \to \mathbb{C}$. Find the matrix of f with respect to the basis $\{1, i\}$.

(27) Let $f\colon V \to V$ be a linear transformation on an n-dimensional vector space with minimal polynomial $m(x) = x^n$.

(i) Show that there is a vector $v \in V$ such that $f^{n-1}(v) \neq 0$.

(ii) Show that $B = \{f^{n-1}(v), f^{n-2}(v), \ldots, f^2(v), f(v), v\}$ is a basis for V.

(iii) Find the matrix for f with respect to the basis B.

(28) Find a linear transformation $f\colon V \to V$ on an infinite dimensional vector space V which satisfies no monic polynomial equation $p(f) = 0$.