Problem sheet 8

Normal subgroups and quotients

Vocabulary

(1) Let G be a group and let H be a subgroup. Define a left coset of H, a right coset of H and the index of H in G and give some illustrative examples.

(2) Let G be a group and let H be a subgroup. Define G/H and give some illustrative examples.

(3) Let G be a group. Define normal subgroup of G and give some illustrative examples.

(4) Let G be a group and let H be a normal subgroup. Define the quotient group G/H and give some illustrative examples.

Results

(1) Let G be a group and let H be a subgroup of G. Let $a, b \in G$. Show that $Ha = Hb$ if and only if $ab^{-1} \in H$.

(2) Let G be a group and let H be a subgroup of G. Show that each element of G lies in exactly one coset of G.

(3) Let G be a group and let H be a subgroup of G. Let $a, b \in G$. Show that the function $f: Ha \to Hb$ given by $f(ha) = hb$ is a bijection.

(4) Let G be a group and let H be a subgroup of G. Show that G/H is a partition of G.

(5) Let G be a group and let H be a subgroup of G. Let $g \in G$. Show that gH and H have the same number of elements.

(6) Let G be a group of finite order and let H be a subgroup of G. Show that $\mathrm{Card}(H)$ divides $\mathrm{Card}(G)$.

(7) Let G be a group of finite order and let $g \in G$. Show that the order of g divides the order of G.

(8) Let G be a finite group and let $n = \mathrm{Card}(G)$. Show that if $g \in G$ then $g^n = 1$.

(9) Let p be a prime positive integer. Show that if a is an integer which is not a multiple of p then $a^{p-1} = 1 \bmod p$.

(10) Let p be a prime positive integer. Let G be a group of order p. Show that G is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

(11) Let G be a group and let H be a subgroup of G. Show that H is a normal subgroup of G if and only if H satisfies if $g \in G$ then $Hg = gH$.

(12) Let G be a group and let H be a subgroup of G. Show that H is a normal subgroup of G if and only if H satisfies if $g \in G$ then $gHg^{-1} = H$.

(13) Let G be a group and let H be a normal subgroup of G. Show that if $ab \in G$ then $HaHb = Hab$.

(14) Let G be a group and let H be a normal subgroup of G. Show that G/H with operation given by $(g_1H)(g_2H) = g_1g_2H$ is a group.

(15) Let $f : G \to G$ be a group homomorphism. Show that $\ker f$ is a normal subgroup of G.

(16) Let $f : G \to H$ be a group homomorphism. Show that im f is a subgroup of H.

(17) Let $f : G \to H$ be a group homomorphism. Show that f is injective if and only if $\ker f = \{1\}$.

(18) Let G be a group and let H be a normal subgroup of G. Let $f : G \to G/H$ be given by $f(g) = gH$. Show that
   (a) f is a group homomorphism,
   (b) $\ker f = H$,
   (c) $im\ f = G/H$.

(19) Let $f : G \to H$ be a group homomorphism. Show that $G/\ker f = \text{im } f$.

Examples and computations

(1) Let
$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
Show that A has order 3, that B has order 4 and that AB has infinite order.

(2) Assume that G is a group such that
$$\text{if } g, h \in G \qquad \text{then} \qquad (gh)^2 = g^2h^2.$$
Show that G is commutative.

(3) Decide whether the positive integers is a subgroup of the integers with operation addition.

(4) Decide whether the set of permutations which fix 1 is a subgroup of S_n.

(5) List all subgroups of $\mathbb{Z}/12\mathbb{Z}$.

(6) Let G be a group, let H be a subgroup and let $g \in G$. Show that $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$ is a subgroup of G.

(7) Let G be a group and let $g \in G$. Let $f\colon G \to G$ be given by $f(h) = ghg^{-1}$. Show that f is an isomorphism.

(8) Show that $SO_2(\mathbb{R})$ is isomorphic to $U_1(\mathbb{C})$.

(9) Show that $(\mathbb{R}, +)$ and $(\mathbb{R}^\times,)$ are not isomorphic.

(10) Show that $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$ are not isomorphic.

(11) Show that $(\mathbb{Z}, +)$ and $(\mathbb{Q}_{>0}, \cdot)$ are not isomorphic.

(12) Show that $SL_2(\mathbb{Z})$ is a subgroup of $GL_2(\mathbb{R})$.

(13) Find the orders of elements 1, -1, 2 and i in the group $\mathbb{C}^\times = \mathbb{C} - \{0\}$ with operation multiplication.

(14) Find the orders of elements in $\mathbb{Z}/6\mathbb{Z}$.

(15) Find the subgroups of $\mathbb{Z}/6\mathbb{Z}$.

(16) Write the element (345) in S_5 in diagram notation, two line notation, and as a permutation matrix, and determine its order.

(17) Write the element (13425) in S_5 in diagram notation, two line notation, and as a permutation matrix, and determine its order.

(18) Write the element (13)(24) in S_5 in diagram notation, two line notation, and as a permutation matrix, and determine its order.

(19) Write the element (12)(345) in S_5 in diagram notation, two line notation, and as a permutation matrix, and determine its order.

(20) Let n be a positive integer. Determine if the group of complex nth roots of unity $\{z \in \mathbb{C} \mid z^n = 1\}$ (with operation multiplication) is a cyclic group.

(21) Determine if the rational numbers $\mathbb{Q}$ with operation addition is a cyclic group.

(22) Find the order of the element $(1, 2)$ in the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$.

(23) Show that the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and the group $\mathbb{Z}/12\mathbb{Z}$ are not isomorphic.

(24) Show that the group $\mathbb{Z} \times \mathbb{Z}$ and the group $\mathbb{Q}$ with operation addition are not isomorphic.

(25) Let G be a group and let $a, b \in G$. Assume that $ab = ba$.
   (a) Prove, by induction, that if $n \in \mathbb{Z}_{>0}$ then $ab^n = b^n a$,
   (b) Prove, by induction, that if $n \in \mathbb{Z}_{>0}$ then $a^n b^n = b^n a^n$,

(c) Show that the order of ab divides the least common multiple of the order of a and the order of b.

(d) Show that if $a = (12)$ and $b = (13)$ then the order of ab does not divide the least common multiple of the order of a and the order of b.

(26) Show that the order of $GL_2(\mathbb{Z}/2\mathbb{Z})$ is 6.

(27) Let p be a prime positive integer. Find the order of the group $GL_2(\mathbb{Z}/p\mathbb{Z})$.

(28) Let $n \in \mathbb{Z}_{>0}$ and let p be a prime positive integer. Find the order of the group $GL_n(\mathbb{Z}/p\mathbb{Z})$.

(29) Show that the group $\mathbb{Z}[x]$ of polynomials with integer coefficients with operation addition is isomorphic to the group $\mathbb{Q}_{>0}$ with operation multiplication.

(30) Let G be a group with less than 100 elements which has subgroups of orders 10 and 25. Find the order of G.

(31) Let G be a group and let H and K be subgroups of G. Show that $|H \cap K|$ is a common divisor of $|H|$ and $|K|$.

(32) Let G be a group and let H and K be subgroups of G. Assume that $|H| = 7$ and $|K| = 29$. Show that $H \cap K = \{1\}$.

(33) Let H be the subgroup of $G = \mathbb{Z}/6\mathbb{Z}$ generated by 3. Compute the right cosets of H in G and the index $|G : H|$.

(34) Let H be the subgroup of $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ generated by $(1, 0)$. Find the order of each element in G/H and identify the group G/H.

(35) Let H be the subgroup of $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ generated by $(0, 2)$. Find the order of each element in G/H and identify the group G/H.

(36) Let $n \in \mathbb{Z}_{\geqslant 2}$ and define $f : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ by $f(A) = A^t$. Determine whether f is a group homomorphism.

(37) Let $n \in \mathbb{Z}_{\geqslant 2}$ and define $f : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ by $f(A) = (A^{-1})^t$. Determine whether f is a group homomorphism.

(38) Let $n \in \mathbb{Z}_{\geqslant 2}$ and define $f : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ by $f(A) = A^2$. Determine whether f is a group homomorphism.

(39) Let B be the subgroup of $GL_2(\mathbb{R})$ of upper triangular matrices and let T be the subgroup of $GL_2(\mathbb{R})$ of diagonal matrices. Let $f : B \to T$ be given by

$$f\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}.$$

Show that f is a group homomorphism. Find $N = \ker f$ and identify the quotient B/N.

(40) Assume G is a cyclic group and let N be a subgroup of G. Show that N is a normal subgroup of G and that G/N is a cyclic group.

(41) Simplify $3^{52} \bmod 53$.

(42) Suppose that $2^{147052} = 76511 \bmod 147053$. What can you conclude about 147053?

(43) Show that if $f\colon G \to H$ is a group homomorphism and $a_1, a_2, \ldots, a_n \in G$ then $f(a_1 a_1 \cdots a_n) = f(a_1)f(a_2)\cdots f(a_n)$.

(44) Describe all group homomorphisms $f\colon \mathbb{Z} \to \mathbb{Z}$.

(45) Show that $SO_n(\mathbb{R})$ is a normal subgroup of $O_n(\mathbb{R})$ by finding a homomorphism $f\colon O_n(\mathbb{R}) \to \{\pm 1\}$ with kernel $SO_n(\mathbb{R})$. Identify the quotient $O_n(\mathbb{R})/SO_n(\mathbb{R})$.

(46) Show that $SU_n(\mathbb{C})$ is a normal subgroup of $U_n(\mathbb{C})$ by finding a homomorphism $f\colon U_n(\mathbb{C}) \to U_1(\mathbb{C})$ with kernel $SU_n(\mathbb{C})$. Identify the quotient $U_n(\mathbb{C})/SU_n(\mathbb{C})$.

(47) Let G be a group and let H be a subgroup of G. Let $f\colon G/H \to H\backslash G$ be given by $f(aH) = Ha^{-1}$.. Show that f is a function and that f is a bijection.

(48) Let $G = \mathbb{Z}$ and $H = 2\mathbb{Z}$. Compute the cosets of H in G and the index $|G : H|$.

(49) Let $G = S_3$ and let H be the subgroup generated by (123). Compute the cosets of H in G and the index $|G : H|$.

(50) Let $G = S_3$ and let H be the subgroup generated by (12). Compute the cosets of H in G and the index $|G : H|$.

(51) Let $G = GL_2(\mathbb{R})$ and let $H = SL2(\mathbb{R})$. Compute the cosets of H in G and the index $|G : H|$.

(52) Let G be the subgroup of $GL_2(\mathbb{R})$ given by

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{R}_{>0},\ y \in \mathbb{R} \right\}$$

Let H be the subgroup of G given by

$$H = \left\{ \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \middle| z \in \mathbb{R}_{>0} \right\}$$

Each element of G can be identified with a point (x, y) of $\mathbb{R}^2$. Use this to describe the right cosets of H in G geometrically. Do the same for the left cosets of H in G.

(53) Consider the set $AX = B$ of linear equations where X and B are column vectors, X is the matrix of unknowns, and A the matrix of coefficients. Let W be the subspace of $\mathbb{R}^n$ which is the set of solutions of the homogeneous equations $AX = 0$. Show that the set of solutions of $AX = B$ is either empty or is a coset of W in the group $\mathbb{R}^n$ (with operation addition).

(54) Let H be a subgroup of index 2 in a group G. Show that if $a, b \in G$ and $a \notin H$ and $b \notin H$.

(55) Let G be a group. Let H be a subgroup of G such that if $a, b \in G$ and $a \notin H$ and $b \notin H$. Show that H has index 2 in G.

(56) Let G be a group of order $841 = (29)^2$. Assume that G is not cyclic. Show that if $g \in G$ then $g^{29} = 1$.

(57) Show that the subgroup $\{(1), (123), (132)\}$ of S_3 is a normal subgroup.

(58) Show that the subgroup $\{(1), (12)\}$ of S_3 is not a normal subgroup.

(59) Show that $SL_n(?)$ is a normal subgroup of $GL_n(\mathbb{C})$.

(60) Let G be a group. Show that $\{1\}$ and G are normal subgroups of G.

(61) Show that every subgroup of an abelian group is normal.

(62) Write down the cosets in $GL_n(\mathbb{C})/SL_n(\mathbb{C})$ then show that $GL_n(\mathbb{C})/SL_n(\mathbb{C})?GL_1(\mathbb{C})$.

(63) Show that the function $\det\colon GL_n(\mathbb{C}) \to GL_1(\mathbb{C})$ given by taking the determinant of a matrix is a homomorphism.

(64) Show that the function $f\colon GL_1(\mathbb{C}) \to GL_1(\mathbb{R})$ given by $f(z) = |z|$ is a homomorphism.

(65) Show that the determinant function $\det\colon GL_n(\mathbb{C}) \to GL_1(\mathbb{C})$ is surjective and has kernel $SL_n(\mathbb{C})$.

(66) Show that the homomorphism $f\colon GL_1(\mathbb{C}) \to GL_1(\mathbb{R})$ given by $f(z) = |z|$ has image $\mathbb{R}_{>0}$ and kernel $U_1(\mathbb{C})$ (the group of 1×1 unitary matrices. Conclude that

$$GL_1(\mathbb{C})/U_1(\mathbb{C}) \cong \mathbb{R}_{>0}.$$

(67) Show that the homomorphism

$$\begin{array}{rccc} f\colon & \mathbb{R} & \longrightarrow & SO_2(\mathbb{R}) \\ & \theta & \longmapsto & \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \end{array}$$

is surjective with kernel $2\pi i\mathbb{Z}$. Conclude that $\mathbb{R}/(2\pi i\mathbb{Z}) \cong SO_2(\mathbb{R})$.

(68) Show that the set of matrices $H = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| ad \neq 0 \right\}$ is a subgroup of $GL_2(\mathbb{R})$ and that the set of matrices $K = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} b \in \mathbb{R} \right\}$ is a normal subgroup of H.

(69) Let G be a group and let H be a subgroup of G. Show that $HH = H$.

(70) Let G be a group and let K and L be normal subgroups of G. Show that $K \cap L$ is a normal subgroup of G.

(71) Let G be a group and let n be a positive integer. Assume that H is the only subgroup of G of order n. Show that H is a normal subgroup of G.

(72) Let G be an abelian group and let N be a normal subgroup of G. Show that G/N is abelian.

(73) Let G be a cyclic group and let N be a normal subgroup of G. Show that G/N is cyclic.

(74) Find surjective homomorphisms from $\mathbb{Z}/8\mathbb{Z}$ to $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ and $\{1\}$ (the group with one element).

(75) Let $\mathbb{R}$ denote the group of real numbers with the operation of addition and let $\mathbb{Q}$ and $\mathbb{Z}$ be the subgroups of rational numbers and integers, respectively. Show that it is possible to regard $\mathbb{Q}/\mathbb{Z}$ as a subgroup of $\mathbb{R}/\mathbb{Z}$ and show that this subgroup consists exactly of the elements of finite order in $\mathbb{R}/\mathbb{Z}$.