Tutorial 10

Main topics: Adjoints, spectral theorem

1. For each of the following matrices decide if it is: (i) self-adjoint (ii) isometric (iii) normal

$$A = \begin{bmatrix} 2 & i \\ -i & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & i \\ 1 & 2+i \end{bmatrix}$$

2. Find a unitary matrix U such that U^*AU is diagonal where $A = \begin{bmatrix} 2 & i \\ -i & 2 \end{bmatrix}$.

3. Let $f\colon V \to V$ be a linear transformation on an inner product space V.
   Show that $\ker(f^*) = \operatorname{im}(f)^\perp$.

4. Let $f\colon V \to V$, where V is a finite-dimensional inner product space.
   (a) Show that if f is self-adjoint, then its eigenvalues are real.
   (Hint: consider $\langle f(v), v\rangle$ where v is an eigenvector)
   (b) Show that if f is an isometry, then its eigenvalues have absolute value 1.
   (Hint: consider $\langle f(v), f(v)\rangle$)

5. Show that every normal matrix A has a square root (i.e., there exists a complex matrix B such that $B^2 = A$).

6. Show that any complex square matrix A can be written uniquely as a sum $A = B + C$ where B is self-adjoint and C is skew self-adjoint (i.e., $C^* = -C$). Further, A is normal if and only if B and C commute (i.e., $BC = CB$).