Tutorial 3

Main topics: Bases, linear transformations, eigenvalues, direct sums, invariant subspaces

1. Consider the subset $S = \{([1]_5, [3]_5), ([3]_5, [4]_5), ([2]_5, [3]_5)\}$ of $\mathbb{F}_5^2$.
   (a) Does S span $\mathbb{F}_5^2$?
   (b) Is S linearly independent?
   (c) Find a subset of S that is a basis for $\mathbb{F}_5^2$.
2. Are the following sets of functions from $\mathbb{R}$ to $\mathbb{R}$ linearly independent?
   (a) $\{1, \sin^2 x, \cos^2 x\}$ (b) $\{1, \sin(2x), \cos(2x)\}$
3. Show that $\{1, \sqrt{2}, \sqrt{3}\}$ is linearly independent over the field of scalars $\mathbb{Q}$.
   (Hint: Is $\sqrt{6}$ rational?)
4. We can regard the complex numbers $\mathbb{C}$ as a vector space V over the field of real numbers $\mathbb{R}$. Note that $\mathcal{B} = \{1, i\}$ is a basis for V over $\mathbb{R}$.
   (a) Let $\alpha = a + ib$ be a complex number. Show that multiplication by α is a linear transformation $f\colon V \to V$.
   (b) Find the matrix of f with respect to the basis $\mathcal{B}$.
   (c) What are the eigenvalues of multiplication by i?
5. Let $f\colon \mathbb{F}_5^4 \to \mathbb{F}_5^4$ be the linear transformation whose matrix in the standard basis is
$$A = \begin{bmatrix} 3 & 4 & 1 & 0 \\ 1 & 1 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$
   Let $U = \mathrm{Span}\{(1, 2, 0, 3), (0, 1, 1, 4)\}$ and $W = \mathrm{Span}\{(1, 2, 2, 0), (4, 0, 1, 3), (2, 1, 2, 3)\}$
   (a) Find bases for U and for W.
   (b) Show that $\mathbb{F}_5^4 = U \oplus W$.
   (c) Check that both U and W are f-invariant.
   (d) Find a block diagonal matrix for f, using an appropriate basis for $\mathbb{F}_5^4$.
6. Consider the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.
   (a) Show that the complex eigenvalues of A are $1, \omega, \omega^2$ where $\omega = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$.
   (b) Find an eigenvector in $\mathbb{C}^2$ corresponding to each eigenvalue.
   (c) Explain why A is diagonalisable over $\mathbb{C}$ and find a diagonal matrix D and an invertible matrix P such that $P^{-1}AP = D$.
   (d) Explain why A is not diagonalisable over $\mathbb{R}$.