Tutorial 9

Main topics: Inner products

1. Explain why the following do *not* define inner products on $\mathbb{C}^2$:
   (a) $\langle (z_1, z_2), (w_1, w_2) \rangle = z_1 w_1 + 4 z_2 w_2$
   (b) $\langle (z_1, z_2), (w_1, w_2) \rangle = z_1 \overline{w_1} - z_2 \overline{w_2}$
   (c) $\langle (z_1, z_2), (w_1, w_2) \rangle = z_1 \overline{w_1}$

2. Find the length of the vector $(1 - 2i, 2 + 3i) \in \mathbb{C}^2$ using the standard inner product on $\mathbb{C}^2$.

3. Find an orthonormal basis for $\mathbb{C}^2$ containing a multiple of the vector $(1 + i, 1 - i)$.
   [Hint: use Gram-Schmidt.]

4. Show that in every complex inner product space V:
$$\forall u, v \in V \quad 4\langle u, v \rangle = \|u + v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - i\|u - iv\|^2$$

5. Let $(\cdot, \cdot)$ be an inner product on a complex vector space V. Let ${}_{\mathbb{R}}V$ be V regarded as a real vector space, and define $\langle v, w \rangle = \operatorname{Re}(v, w)$.
   (a) Show that $\langle v, w \rangle$ is an inner product on ${}_{\mathbb{R}}V$.
   (b) Show that: $\forall v, w \in V \quad (v, w) = \langle v, w \rangle + i\langle v, iw \rangle$
   (c) Deduce that $(v, w) = 0$ if and only if $\langle v, w \rangle = 0$ and $\langle v, iw \rangle = 0$.

6. Let W be the subspace of $\mathbb{R}^4$ spanned by the set $\{(0, 1, 0, 1), (2, 0, -3, -1)\}$. Find a basis for the orthogonal complement $W^\perp$ (where inner product is the usual dot product).
   [Hint: $x \in W^\perp$ if and only if $x \cdot (0, 1, 0, 1) = 0$ and $x \cdot (2, 0, -3, -1) = 0$.]

7. Let W be a subspace of an inner product space V. Show that $W \subset (W^\perp)^\perp$ and that $W = (W^\perp)^\perp$ if $\dim V$ is finite.