GTLA Lecture 15.09.2020

Let G be a group and H a subgroup.

The <u>set of cosets of H in G</u> is

$G/H = \{gH \mid g \in G\}$ where

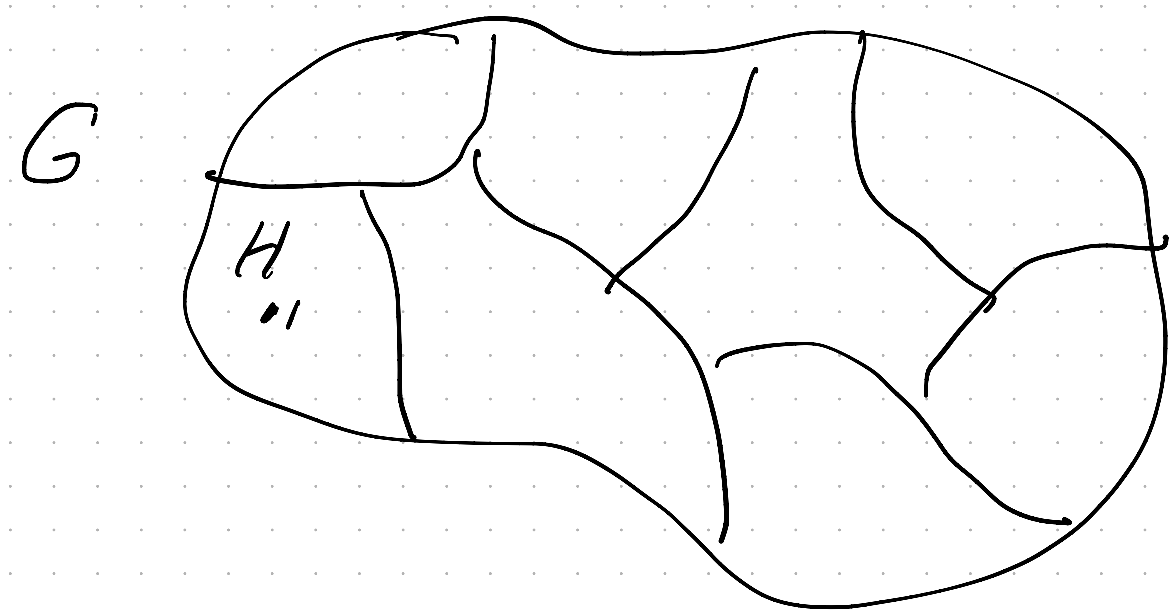
$gH = \{gh \mid h \in H\}$.

<u>Theorem</u> $\mathrm{Card}(G) = \mathrm{Card}(G/H)\,\mathrm{Card}(H)$,

even better

the cosets partition G and

$\mathrm{Card}(gH) = \mathrm{Card}(H)$



Can we multiply cosets?

Proposition Let G be a group and H a subgroup. Then

$$G/H \times G/H \to G/H$$
$$(aH, bH) \mapsto abH$$

~~makes a group~~ is a function if and only if H is normal.

~~WRONG!~~ **RIGHT!**

Example $G = S_3 = D_3 = \{1, r, r^2, s, sr, sr^2\}$

with $r^3 = 1$, $s^2 = 1$, $rs = sr^{-1}$.

Let $H = \{1, s\}$, $s^2 = 1$.

The cosets of H:

$H = \{1, s\}$ $= sH$, $rH = \{r, rs\}$ $= rsH$, $r^2H = \{r^2, r^2s\}$ $= r^2sH$

So $G/H = \{H, rH, r^2H\}$.

Let $\mu : G/H \times G/H \to G/H$
$(aH, bH) \mapsto abH$.

Then

$$\mu(1H, rH) = 1\cdot rH = rH = \{r, rs\}$$

$$\parallel$$

$$= \mu(sH, rH) = srH = r^2sH = \{r^2, r^2s\}$$

$\neq$

OOPS. So μ is not a function

(The input $(H, rH) = (sH, rH)$ does not have a unique output).

This is just like

$$\mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$$
$$x \longmapsto \sqrt{x}$$

is not a function.

since $\sqrt{9} = 3$ and $\sqrt{9} = -3$.

and $3 \neq -3$.

A subgroup H is normal if it satisfies:

If $h \in H$ and $g \in G$ then $g^{-1}hg \in H$.

Proof $\Rightarrow$ Assume μ is a function.

To show: H is normal.

To show: If $h \in H$ and $g \in G$ then
$g^{-1}hg \in H$.

Assume $h \in H$ and $g \in G$.

To show: $g^{-1}hg \in H$.

Since $g^{-1}hH = g^{-1}H$,

$$g^{-1}hg \in g^{-1}hgH = \mu(g^{-1}hH, gH)$$
$$= \mu(g^{-1}H, gH)$$
$$= g^{-1}gH = 1 \cdot H = H.$$

So $g^{-1}hg \in H$.

So H is normal.

$\Leftarrow$ Assume H is normal.

To show: μ is a function.

To show: If $a_1, a_2, b_1, b_2 \in G$ and
$a_1H = a_2H$ and $b_1H = b_2H$
then $a_1b_1H = a_2b_2H$.

$\left(a_1b_1H = \mu(a_1H, b_1H) \text{ and } a_2b_2H = \mu(a_2H, b_2H) \right)$

Assume $a_1, a_2, b_1, b_2 \in G$ and $a_1H = a_2H$ and $b_1H = b_2H$.

Since $a_1 \in a_2H$ then there exists $h_1 \in H$ such that $a_1 = a_2h_1$

Since $b_1 \in b_2H$ then there exists $h_2 \in H$ such that $b_1 = b_2h_2$

Then

$$a_1b_1 = a_2h_1b_2h_2 = a_2b_2b_2^{-1}h_1b_2h_2$$
$$= a_2b_2\underbrace{(b_2^{-1}h_1b_2)h_2} \in a_2b_2H$$

since $b_2^{-1}h_1b_2 \in H$ (because H is normal)

and $(b_2^{-1}h_1b_2)h_2 \in H$ (because H is a subgroup).

So $a_1b_1H \cap a_2b_2H \neq \emptyset$.

So $a_1b_1H = a_2b_2H$, since the cosets partition G.

So μ is a function. //

Proposition If G is a group and H is a subgroup then

$$G/H \times G/H \xrightarrow{\mu} G/H \qquad \text{is a function}$$
$$(aH, bH) \mapsto abH$$

if and only if H is normal.

Theorem If μ is a function then G/H is a group.

Proof To show:

(a) If $a_1, a_2, a_3 \in G$ then
$$a_1H \cdot (a_2H \cdot a_3H) = (a_1H \cdot a_2H) \cdot a_3H.$$

(b) There exists $e \in G$ such that if $g \in G$ then
$$eH \cdot gH = gH \text{ and } gH \cdot eH = gH.$$

(c) If $g \in G$ then there exists $b \in G$ such that
$$gH \cdot bH = eH \text{ and } bH \cdot gH = eH.$$

(a) Assume $a_1, a_2, a_3 \in G$. Then

$a_1H(a_2H \cdot a_3H) = a_1H(a_2a_3H)$

$= a_1(a_2a_3)H$ and

$(a_1H \cdot a_2H) \cdot a_3H = a_1a_2H \cdot a_3H$

$= (a_1a_2)a_3H$. and

$a_1(a_2a_3) = (a_1a_2)a_3$ by associativity in G. So G/H is associative.

(b) To show: There exists $e \in G$ such that if $g \in G$ then

$eH \cdot gH = gH$ and $gH \cdot eH = gH$.

Let $e = 1$, where 1 is the identity in G

To show: ~~If~~ $g \in G$ then

$eH \cdot gH = gH$ and $gH \cdot eH = gH$.

Assume $g \in G$. Then

$eH \cdot gH = egH = 1 \cdot gH = gH$ and

$gH \cdot eH = geH = g \cdot 1H = gH$.

So $eH = 1 \cdot H = H$ is the identity in G/H.

(c) Let $g \in G$.

To show: There exists $b \in G$ such that

$$bH \cdot gH = eH \text{ and } gH \cdot bH = eH.$$

Let $b = g^{-1}$, the inverse of g in G.

Then

$$bH \cdot gH = bgH = g^{-1}gH = 1 \cdot H = eH \text{ and}$$

$$gH \cdot bH = gbH = gg^{-1}H = 1 \cdot H = eH.$$

So $g^{-1}H$ is the inverse of gH in G/H.

(also $g^{-1}hH$ is the inverse of gH in G/H

BUT $g^{-1}h$ is <u>not</u> the inverse of g in G)

//.

Definition Let G be a group and let N be a normal subgroup. The quotient group is G/N with

$$G/N \times G/N \longrightarrow G/N$$
$$(aN, bN) \longmapsto abN.$$

Now we've finally explained why clocks are denoted $\mathbb{Z}/m\mathbb{Z}$. (Let $m \in \mathbb{Z}_{>0}$).

$\mathbb{Z}$ is a group (under $+$)

$m\mathbb{Z}$ is a normal subgroup.

$\mathbb{Z}/m\mathbb{Z} = \{0+m\mathbb{Z}, 1+m\mathbb{Z}, \dots, m-1+m\mathbb{Z}\}$ is the set of cosets.

$$0+m+m\mathbb{Z} = 0+m\mathbb{Z}$$

So $m+m\mathbb{Z} = 0+m\mathbb{Z}$

Just as $12=0$ in $\mathbb{Z}/12\mathbb{Z}$.

Since
If $b \in m\mathbb{Z}$ and $a \in \mathbb{Z}$

then $(-a)+b+a \in m\mathbb{Z}$

then $m\mathbb{Z}$ is normal.

"Grace's theorem"
If $g \in G$ and $h \in H$

then $ghH = gH$.