<u>GTLA Lecture 29.09.2020</u>

Let $\mathbb{F}$ be a field with $\bar{\ }:\mathbb{F}\to\mathbb{F}$ such that $\bar{\bar{c}}=c$, $\overline{c_1+c_2}=\bar{c}_1+\bar{c}_2$, $\overline{c_1c_2}=\bar{c}_1\bar{c}_2$ and $\bar{1}=1$.

Let V be an $\mathbb{F}$-vector space.
Let $\langle\,,\rangle : V\times V\to\mathbb{F}$ be a sesquilinear form.
Let $W\subseteq V$ a subspace. Let $k\in\mathbb{Z}_{>0}$ be $\dim(W)=k$.
Let $\{w_1,\dots,w_k\}$ be a basis of W.

The <u>dual basis to $\{w_1,\dots,w_k\}$</u> is $\{w^1,\dots,w^k\}$ such that

$$\langle w^i, w_j\rangle=\delta_{ij}.$$

<u>Proposition</u> The following are equivalent:

(1) $\{w^1,\dots,w^k\}$ exists

(2) The Gram matrix G is invertible.

$(G(i,j) = \langle w_i, w_j \rangle)$.

(3) $W \cap W^\perp = 0$.

Proof (2) $\Leftrightarrow$ (3). (Sketch).

Let $w \in W \cap W^\perp$. Let $c_1, \ldots, c_k \in \mathbb{F}$ such that

$$w = c_1 w_1 + \cdots + c_k w_k$$

Since $w \in W^\perp$ then

$$0 = \langle w, w_i \rangle = \sum_{j=1}^{k} \langle c_j w_j, w_i \rangle$$

$$= \sum_{j=1}^{k} c_j \langle w_j, w_i \rangle$$

$$= \sum_{j=1}^{k} c_j G(j, i)$$

$=$ ith entry of $(c_1, \ldots, c_k) \Big(\quad G \quad \Big)$

So

$$(0,\dots,0) = (c_1,\dots c_k)\begin{pmatrix} - & G_1 & - \\ - & G_2 & - \\ & \vdots & \\ - & G_k & - \end{pmatrix}$$

The rows of G are linearly independent
$\iff$ For all choices of $w \in W \cap W^\perp$
$c_1 = 0, \dots, c_k = 0$
$\iff$ For all choices of $w \in W \cap W^\perp$
$w = 0$.
$(w = c_1 w_1 + \cdots + c_k w_k)$.

So G is invertible

$\iff W \cap W^\perp = 0$. //

Orthonormal bases

Assume $\langle , \rangle$ is Hermitian,

i.e. if $v_1, v_2 \in V$ then $\langle v_2, v_1 \rangle = \overline{\langle v_1, v_2 \rangle}$

An <u>orthonormal basis</u> <u>of W</u>, or <u>selfdual basis</u>, is $(u_1, u_2, \dots, u_k)$ such that

$$\langle u_i, u_j \rangle = \delta_{ij}.$$

Assume $W \cap W^\perp = 0$

or if $v \in V$ and $\langle v, v \rangle = 0$ then $v = 0$.

Construct orthonormal bases with <u>Gram-Schmidt process</u>.

Let $\{w_1, \dots, w_k\}$ be a basis of W.

$C_1 = \{b_1, w_2, w_3, \dots, w_k\}$ with $b_1 = w_1$

$C_2 = \{b_1, b_2, w_3, w_4, \dots, w_k\}$ with

$$b_2 = w_2 - \frac{\langle w_2, b_1\rangle}{\langle b_1, b_1\rangle} b_1$$

(so that

$$\langle b_2, b_1\rangle = \left\langle w_2 - \frac{\langle w_2, b_1\rangle}{\langle b_1, b_1\rangle} b_1, \; b_1\right\rangle$$

$$= \langle w_2, b_1\rangle - \frac{\langle w_2, b_1\rangle}{\cancel{\langle b_1, b_1\rangle}} \cancel{\langle b_1, b_1\rangle}$$

$$= 0$$
)

$C_3 = \{b_1, b_2, b_3, w_4, w_5, \dots, w_k\}$ with

$$b_3 = w_3 - \frac{\langle w_3, b_1\rangle}{\langle b_1, b_1\rangle} b_1 - \frac{\langle w_3, b_2\rangle}{\langle b_2, b_2\rangle} b_2 .$$

(so that $\langle b_3, b_2\rangle = 0$ and $\langle b_3, b_1\rangle = 0$).

⋮

$C_K = \{b_1, b_2, \dots, b_K\}$.

This has $\langle b_j, b_i \rangle = 0$ if $i < j$.

and $\langle b_i, b_j \rangle = \overline{\langle b_j, b_i \rangle} = 0$.

So C_K is orthogonal.

Assume that $\mathbb{F}$ has good square roots.

(in most cases one assumes $\langle v, v \rangle \in \mathbb{R}_{\geq 0}$ and $\langle v, v \rangle \neq 0$ if $v \neq 0$)

Let $U = \{u_1, \dots, u_K\}$ given by

$$u_1 = \frac{b_1}{\sqrt{\langle b_1, b_1 \rangle}}, \quad \dots, \quad u_K = \frac{b_K}{\sqrt{\langle b_K, b_K \rangle}}$$

Then $\langle u_i, u_i \rangle = 1$

and $\langle u_i, u_j \rangle = 0$.

$\Big\}$ orthonormal.

Gram-Schmidt process

Start with $\{w_1, \dots, w_k\}$.
Recursively construct $b_1, \dots, b_k$ to get

$\{b_1, \dots, b_k\}$ orthogonal basis.

Make the vectors unit vectors to get $\{u_1, \dots, u_k\}$ orthonormal basis.

Example $w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $w_2 = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$

$\{w_1, w_2\}$ is basis (not orthogonal)

Let

$$b_1 = w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$b_2 = w_2 - \frac{\langle w_2, b_1 \rangle}{\langle b_1, b_1 \rangle} b_1 = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} - \frac{2+0+4}{1+1+1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}.$$

$(b_2 \cdot b_1 = 0)$

So $\{b_1, b_2\}$ are orthogonal.

Let $u_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{pmatrix}$

$(b_2 \cdot b_2 = 4+4$ so $\sqrt{b_2 \cdot b_2} = \sqrt{2}\cdot 2)$

$$u_2 = \frac{1}{\sqrt{2}\cdot 2}\begin{pmatrix}0\\-2\\2\end{pmatrix} = \begin{pmatrix}0\\-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

Then $\{u_1, u_2\}$ is an orthonormal basis of W.

<u>To show</u>: (2) $\Leftrightarrow$ (3) where

(2) G is invertible

(3) $W \cap W^\perp = 0$.

$\Rightarrow$: Assume G is invertible. So the rows of G are linearly independent.

To show: If $w \in W \cap W^\perp$ then

$w = 0$.

Assume $w \in W \cap W^\perp$.

Let $c_1, \ldots, c_k \in \mathbb{F}$ be such that

$$w = c_1 w_1 + \cdots + c_k w_k.$$

Since $w \in W^\perp$ then

$0 = \langle w_1, w \rangle, \quad \ldots, \quad 0 = \langle w_k, w \rangle$

So

$$(0, \ldots, 0) = (c_1, \ldots, c_k) \begin{pmatrix} & G & \end{pmatrix}$$

So

$$(0, \ldots, 0) \begin{pmatrix} G^{-1} \end{pmatrix} = (c_1, \ldots, c_k)$$

So

$$(0, \ldots, 0) = (c_1, \ldots, c_k)$$

So $w = 0 w_1 + \cdots + 0 w_k = 0$.

So $W \cap W^\perp = 0$.

$\Leftarrow$ Assume $W \cap W^\perp = 0$.

To show: G is invertible.

To show: The rows of G are linearly independendent.

To show: If $c_1, \dots, c_k \in \mathbb{F}$ and

$$(0, \dots 0) = (c_1, \dots, c_k)\begin{pmatrix} & G & \end{pmatrix}$$

then $c_1 = 0, \dots, c_k = 0$.

Assume $c_1, \dots, c_k \in \mathbb{F}$. and

$$(0, \dots, 0) = (c_1 \cdots c_k)\begin{pmatrix} & G & \end{pmatrix}$$

Let $w = c_1 w_1 + \cdots + c_k w_k$.

Then $w \in W$ and

$$\langle w_1, w\rangle = 0, \dots, \langle w_k, w\rangle = 0$$

(because $\langle w_i, w\rangle$ is the ith entry of $(c_1 \cdots c_k)\begin{pmatrix} & G & \end{pmatrix}$)

So $w \in W \cap W^\perp$.

Since $W \cap W^\perp = 0$ then $w = 0$

and $c_1 = 0, \dots, c_k = 0$.