GTLA Lecture 01.10.2020

Let $\langle\, ,\rangle : V \times V \to \mathbb{F}$ be a sesquilinear form. Let W be a finite dimensional subspace of V and $W \cap W^{\perp} = 0$

The <u>orthogonal projection</u> <u>onto W</u> is the unique linear transformation $P_W : V \to V$ such that

(1) If $v \in V$ then $P_W(v) \in W$.

(2) If $v \in V$ and $w \in W$ then

$$\langle v, w\rangle = \langle P_W(v), w\rangle.$$

(A) If $w \in W$ then $w - P_W(w) \in W$ and $w - P_W(w) \in W^{\perp}$ (since if $w' \in W$ then

$$\langle w - P_W(w), w'\rangle = \langle w, w'\rangle - \langle P_W(w), w'\rangle$$
$$= \langle w, w'\rangle - \langle w, w'\rangle = 0.)$$

So

$w - P_W(w) \in W \cap W^\perp = 0.$

So $w - P_W(w) = 0$ and

$P_W(w) = w.$

(B) If $v \in V$ then $P_W(v) \in W$

$P_W(P_W(v)) = P_W(v)$

So $P_W^2 = P_W$ (idempotent).

(C) Define $P_{W^\perp} = 1 - P_W.$

If $v \in V$ then $P_{W^\perp}(v) \in W^\perp$

(since $w' \in W$ then

$\langle P_{W^\perp}(v), w' \rangle = \langle (1 - P_W)(v), w' \rangle$

$= \langle v - P_W(v), w' \rangle$

$= \langle v, w' \rangle - \langle P_W(v), w' \rangle$

$= \langle v, w' \rangle - \langle v, w' \rangle = 0$)

(D) $(P_{W^\perp})^2 = P_{W^\perp}$, ($P_{W^\perp}$ is also idemp.)

since

$(P_{W^\perp})^2 = (1-P_W)(1-P_W)$

$= 1 - 2P_W + P_W^2$

$= 1 - 2P_W + P_W = 1 - P_W = P_{W^\perp}.$

(E) $P_W P_{W^\perp} = 0 = P_{W^\perp} P_W.$

(since

$P_W P_{W^\perp} = P_W(1-P_W)$

$= P_W - P_W^2 = P_W - P_W = 0$)

(F) $1 = P_W + P_{W^\perp}$

since $P_W + P_{W^\perp} = P_W + (1-P_W) = 1.$

<u>Theorem</u> Keep W is a fin. dim'l subspace of V and

$W \cap W^\perp = 0.$

Then

$V = W \oplus W^\perp.$

(orthogonal decomposition).

Proof To show: $V = W \oplus W^\perp$.

To show: (a) $V = W + W^\perp$

(b) $W \cap W^\perp = 0$.

(b) is true by assumption.

(a) To show: (aa) $W + W^\perp \subseteq W$

(ab) $V \subseteq W + W^\perp$.

(aa) Since W and $W^\perp$ are contained in V and V is closed under addition then

$$W + W^\perp \subseteq V.$$

(ab) To show: If $v \in V$ then there exist $x \in W$ and $y \in W^\perp$ such that $v = x + y$.

Assume $v \in V$.

Let $x = P_W(v)$ and $y = P_{W^\perp}(v)$.

Then

$$x + y = P_W(v) + P_{W^\perp}(v)$$

$$= (P_W + P_{W^\perp})(v) = Iv$$

$$= v.$$

So $V \subseteq W + W^\perp$.

So $V = W + W^\perp$ and $V = W \oplus W^\perp$ //.

<u>Remark</u> In Block decomposition theorem we had

$$p(x)r(x) + q(x)s(x) = 1$$

and $P_U = p(A)r(A)$ and

$$P_W = q(A)s(A)$$

and $P_U^2 = P_U$, $P_W^2 = P_W$, $I = P_U + P_W$

and $U \oplus W = V$.

<u>Adjoints</u>: Same setup:

W is a fin. dim'l subspace of V and $W \cap W^\perp = 0$.

Let $\{w_1, \dots, w_k\}$ be a basis of W and $\{w^1, \dots, w^k\}$ be the dual basis to $\{w_1, \dots, w_k\}$ with respect to $\langle , \rangle$, i.e.

$$\langle w^i, w_j \rangle = \delta_{ij}.$$

Let $f: W \to W$ be a linear transformation.

The <u>adjoint of f</u> is the linear transformation $f^*: W \to W$ such that

if $x, y \in W$ then

$$\langle f(x), y \rangle = \langle x, f^*(y) \rangle.$$

If $w \in W$ then write

$$w = c_1 w_1 + \cdots + c_k w_k.$$

Then

$$c_j = \langle w^j, c_j w_j \rangle$$
$$= \langle w^j, c_1 w_1 + \cdots + c_k w_k \rangle$$

$= \langle w^j, w \rangle$

So $w = \overline{\langle w^1, w \rangle} w_1 + \cdots + \overline{\langle w^k, w \rangle} w_k$.

Apply this to $f^*(y)$.

$f^*(y) = \overline{\langle w^1, f^*(y) \rangle} w_1 + \cdots + \overline{\langle w^k, f^*(y) \rangle} w_k$

$= \sum_{i=1}^{k} \overline{\langle w^i, f^*(y) \rangle} w_i$.

So

$$\boxed{f^*(y) = \sum_{i=1}^{k} \overline{\langle f(w^i), y \rangle} w_i}$$

This is a formula for f^* purely in terms of f and $\{w_1, \ldots, w_k\}$ and $\{w^1, \ldots, w^k\}$.

A linear transformation $f: W \to W$ is

(a) <u>self adjoint</u> if $f = f^*$.

(i.e. if $x, y \in W$ then $\langle f(x), y \rangle = \langle x, f(y) \rangle$.)

(b) an <u>isometry</u> if $f^*f = 1$

(i.e. if $x, y \in W$ then $\langle f(x), f(y) \rangle = \langle x, y \rangle$)

$= \langle x, f^*f(y) \rangle$

(c) is <u>normal</u> if $ff^* = f^*f$.

<u>Favorite example</u>

$W = \mathbb{C}^n$ and $\langle , \rangle$ is the standard dot product.

Let $\{e_1, \dots, e_n\}$ be standard basis

$$e_i = \begin{array}{c} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ \\ i^{\text{th}} \\ \\ \\ \\ \end{array}$$

$\{e_1, \dots, e_n\}$ is orthonormal.

Let $f : W \to W$ be a linear transformation. Let $A \in M_n(\mathbb{C})$ be the matrix of f with respect to $\{e_1, \dots, e_n\}$.

So $f: \mathbb{C}^n \longrightarrow \mathbb{C}^n$

$w \longmapsto Aw.$

Claim:

The matrix of $f^*: \mathbb{C}^n \longrightarrow \mathbb{C}^n$

$v \longmapsto A^* v$

(with respect to $\{e_1, \dots, e_n\}$) is

$A^* \in M_n(\mathbb{C})$ given by

$$A^*(i,j) = \overline{A(j,i)}.$$

(we write $A^* = \overline{A}^t$).

since

$$\sum_{l=1}^{n} A^*(l,i) e_l = A^* e_i = f^*(e_i)$$

$$= \sum_{l=1}^{n} \overline{\langle f(e_l), e_i \rangle} e_l. \quad \left(\text{by the formula for } f^*(y)\right)$$

$$= \sum_{l=1}^{n} \overline{\langle Ae_l, e_i \rangle} e_l.$$

$$= \sum_{l=1}^{n} \sum_{k=1}^{n} \overline{\langle A(k,l) e_k, e_i \rangle} e_l$$

$$\overline{\langle c e_k, e_l \rangle} = \overline{c \langle e_k, e_l \rangle} = \bar{c}\, \overline{\langle e_k, e_l \rangle}$$

$$= \sum_{l,k=1}^{n} \overline{A(k,l)} \, \overline{\langle e_k, e_i \rangle} \, e_l .$$

$$= \sum_{l=1}^{n} \overline{A(i,l)} \, e_l .$$

So $A^*(l,i) = \overline{A(i,l)}$.