GTLA Lecture 20.10.2020

The **quaternion group** is

$$Q = \{1, -1, i, -i, j, -j, k, -k\}$$

with $Q \times Q \to Q$ given by

	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	i	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
k	k	-k	j	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

$i^2 = -1$, $j^2 = -1$, $k^2 = -1$, $ij = k$

$jk = i$, $ki = j$

Group of cardinality 8.

order$(1) = 1$ order$(i) = 4$

order$(-1) = 2$ order$(-i) = 4$

order$(j) = 4$, order$(k) = 4$,
order$(-j) = 4$, order$(-k) = 4$,

Q is <u>not</u> isomorphic to D_4

$D_4 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$
with $r^4 = 1$, $s^2 = 1$, $rs = sr^{-1}$

The group $\mu_4 = \{1, i, -1, -i\}$ is the group of 4^{th} roots of 1 in $\mathbb{C}$,

$\mu_4 = \{1, i, i^2, i^3\}$ with $i^4 = 1$.

The <u>complex numbers</u> is

$\mathbb{C} = \mathbb{R}\text{-span}\{1, i, -1, -i\}$
$= \{x + yi \mid x, y \in \mathbb{R}\}$

with multiplication given by the multiplication in μ_4 and $\mathbb{R}$ and the distributive law.

<u>HW</u>: Show that $\mathbb{C}$ is a commutative ring.

Show that $\mathbb{C}$ is a field!

The <u>quaternions</u>, or <u>Hamiltonians</u>

is $\mathbb{H} = \mathbb{R}\text{-span}\{1, -1, i, -i, j, -j, k, -k\}$

$= \{t + xi + yj + zk \mid t, x, y, z \in \mathbb{R}\}$.

with multiplication determined by the multiplication in Q and $\mathbb{R}$ and the distributive law.

<u>HW</u> Show that $\mathbb{H}$ is a <u>noncommutative</u> ring.

For example,

$ij = k$ and $ji = -k$.

Let

$u_1(\mathbb{H}) = \{xi + yj + zk \mid x, y, z \in \mathbb{R}\}$

(maybe call it $\mathbb{R}^3 = u_1(\mathbb{H})$).

Define

$$u_1(\mathbb{H}) \times u_1(\mathbb{H}) \to \mathbb{R}$$
$$(v_1, v_2) \mapsto v_1 \cdot v_2$$

$$u_1(\mathbb{H}) \times u_1(\mathbb{H}) \to u_1(\mathbb{H})$$
$$(v_1, v_2) \mapsto v_1 \times v_2$$

given by

$$(x_1 i + y_1 j + z_1 k)\cdot(x_2 i + y_2 j + z_2 k)$$
$$= x_1x_2 + y_1y_2 + z_1z_2.$$

$$(x_1 i + y_1 j + z_1 k)\times(x_2 i + y_2 j + z_2 k)$$
$$= \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1z_2 - y_2z_1)i$$
$$- (x_1z_2 - z_1x_2)j$$
$$+ (x_1y_2 - y_1x_2)k.$$

__Proposition__ The multiplication $\mathbb{H}$ is given by

$$(t_1+v_1)(t_2+v_2) = (t_1t_2 - v_1\cdot v_2).$$
$$+ (t_1v_2 + t_2v_1$$
$$+ \; v_1 \times v_2)$$

where $t_1, t_2 \in \mathbb{R}$, $v_1, v_2 \in \mathcal{I}_m(\mathbb{H})$.

__Proof__ $t_1 + v_1 = t_1 + x_1 i + y_1 j + z_1 k$
$t_2 + v_2 = t_2 + x_2 i + y_2 j + z_2 k.$

$$(t_1+v_1)(t_2+v_2) = t_1t_2 + t_1v_1 + t_2v_2$$
$$+ v_1v_2$$

$$= t_1 t_2 + t_1 v_2 + t_2 v_1$$
$$+ (x_1 i + y_1 j + z_1 k)(x_2 i + y_2 j + z_2 k)$$
$$= t_1 t_2 + t_1 v_2 + t_2 v_1$$
$$+ x_1 x_2 i^2 + y_1 y_2 j^2 + z_1 z_2 k^2$$
$$+ x_1 y_2 ij + y_1 x_2 ji$$
$$+ x_1 z_2 ik + z_1 x_2 ki$$
$$+ y_1 z_2 jk + z_1 y_2 kj.$$
$$= t_1 t_2 + t_1 v_2 + t_2 v_1$$
$$- (x_1 x_2 + y_1 y_2 + z_1 z_2)$$
$$+ (x_1 y_2 - y_1 x_2) k$$
$$- (x_1 z_2 - z_1 x_2) j$$
$$+ (y_1 z_2 - z_1 y_2) i$$
$$= t_1 t_2 + t_1 v_2 + t_2 v_1 - v_1 \cdot v_2$$
$$+ v_1 \times v_2. \quad //$$

In $\mathbb{C}$ we have

$$\bar{\ } : \mathbb{C} \to \mathbb{C} \qquad | \ | : \mathbb{C} \to \mathbb{R}_{\geq 0}$$
$$x + iy \mapsto x - iy \qquad x + iy \mapsto \sqrt{x^2 + y^2}.$$

In $\mathbb{H}$ we have $\bar{\ }: \mathbb{H} \to \mathbb{H}$

$$\overline{t+xi+yj+zk} = t-xi-yj-zk.$$

and $\| \ \| : \mathbb{H} \to \mathbb{R}_{\geq 0}$ given by

$$\|t+xi+yj+zk\| = \sqrt{t^2+x^2+y^2+z^2}$$

Theorem Let $h \in \mathbb{H}$.

(a) $h\bar{h} = \|h\|^2$.

(b) If $h \neq 0$ then there exists $h^{-1} \in \mathbb{H}$ such that

$$h h^{-1} = h^{-1} h = 1.$$

This says that $\mathbb{H}$ is a "noncommutative field"

Proof Let $h = t + v$, with $t \in \mathbb{R}$ and $v \in \mathrm{Im}(\mathbb{H})$.

i.e. $v = xi+yj+zk$.

Then $\bar{h} = t - v$. So

$$h\bar{h} = (t+v)(t-v)$$

$$= t^2 - tv + tv + v \times (-v) - v \cdot (-v)$$

$= t^2 - \underbrace{v \times v}_{0} + v \cdot v$

$= t^2 + x^2 + y^2 + z^2 = \|h\|^2.$

Assuming $h \neq 0$ so that $\|h\| \neq 0$ then

So let $h^{-1} = \frac{1}{\|h\|^2}\bar{h}$ so that

$h h^{-1} = h\left(\frac{1}{\|h\|^2}\bar{h}\right) = \frac{1}{\|h\|^2} h\bar{h} = \frac{1}{\|h\|^2}\|h\|^2$

$= 1.$

and

$h^{-1}h = \frac{1}{\|h\|^2}\bar{h}\, h = \frac{1}{\|h\|^2}\bar{h}\,\bar{\bar{h}}$

$= \frac{1}{\|h\|^2}\|\bar{h}\|^2 = \frac{1}{\|h\|^2}\|h\|^2 = 1.$

So every nonzero element is invertible. //.

Polar form for elements of $\mathbb{H}$

If $x+yi \in \mathbb{C}$ then

$x + iy = re^{i\theta}$

$= r(\cos\theta + i\sin\theta)$

$= r\cos\theta + (r\sin\theta)i.$

Let $w = ai + bj + ck$ with

$$a^2 + b^2 + c^2 = 1.$$

Then

$$w^2 = (0 + w)^2 = \cancelto{0}{0^2} + \cancelto{0}{0\cdot w} + \cancelto{0}{0\cdot w} - w\cdot w + \cancelto{0}{w \times w}$$

$$= -w\cdot w = -(a^2+b^2+c^2) = -1.$$

Then, if $\theta \in \mathbb{R}$ then

$$e^{w\theta} = \cos\theta + w\sin\theta$$

$$\left(e^{w\theta} = 1 + w\theta + \frac{1}{2!}(w\theta)^2 + \frac{1}{3!}(w\theta)^3 + \cdots \right.$$
$$\left. = \cdots\cdots = \cos\theta + w\sin\theta. \right)$$

If $r \in \mathbb{R}_{\geq 0}$ then

$$re^{w\theta} = r\cos\theta + r\sin\theta\, w$$

$$= r\cos\theta + r\sin\theta \cdot a\, i + r\sin\theta\, b\, j + r\sin\theta\, c\, k.$$

$$= t + xi + yj + zk \in \mathbb{H}.$$

The conversion for this "polar form"

are

$$r = \sqrt{t^2+x^2+y^2+z^2}$$

$$\cos\theta = \frac{t}{r}$$

$$\sin^2\theta = \frac{x^2+y^2+z^2}{r^2}$$

$$a = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$b = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$c = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$t = r\cos\theta$$

$$x = ar\sin\theta$$

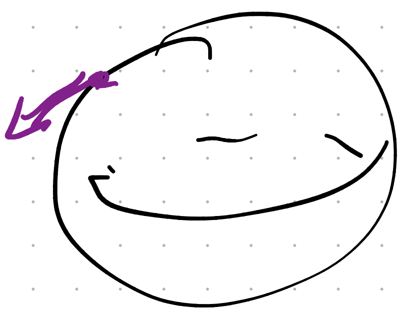
$$y = br\sin\theta$$

$$z = cr\sin\theta$$

$$U_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid A\bar{A}^t = 1\}.$$

$$U_n(\mathbb{H}) = \{A \in M_n(\mathbb{H}) \mid A\bar{A}^t = 1\}.$$

$u_1(\mathbb{H})$ = tangent vectors to $U_1(\mathbb{H})$.



Assume G is a group and
G has two conjugacy classes
To show: (a) G has two elements
(b) G is cyclic.

(b)

$$\{1, g\} \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z}$$
$$1 \mapsto 0$$
$$g \mapsto 1.$$

(a)

$$\mathcal{C}_1 = \{g1g^{-1} \mid g \in G\} = \{1\}.$$

So if $g \neq 1$ then $\mathcal{C}_g = \mathcal{C}_1^c$.

Case 1 $\mathrm{Card}(G) = 3$.

$\{1, h, k\}$

$$\mathcal{C}_h = \{1h1^{-1}, hhh^{-1}, hkh^{-1}\}$$
$$= \mathcal{C}_k = \{1k1^{-1}, kkk^{-1}, khk^{-1}\}.$$

$\mathrm{Card}(G) = n$.

When does
$n-1$ divide n?