GTLA Lecture 22.09.2020

The conjugation action of G on G and "the class equation".

Let G be a group.
The <u>conjugation action of G on G</u>

$$G \times G \longrightarrow G$$
$$(g, x) \longmapsto g \diamond x \quad \text{where}$$

$$g \diamond x = g x g^{-1}.$$

Let $x \in G$. The <u>centralizer of x</u> is

$$\begin{aligned} Z_G(x) &= \{ g \in G \mid g x g^{-1} = x \} \\ &= \{ g \in G \mid g x = x g \} \\ &= \{ g \in G \mid g \diamond x = x \} \\ &= \mathrm{Stab}_G(x) \end{aligned}$$

The <u>conjugacy class of x</u> is

$$\mathcal{C}_x = \{ g x g^{-1} \mid g \in G \} = G \diamond x$$

So $\mathcal{C}_x$ is the orbit of x and $Z_G(x)$ is the stabilizer of x under the conjugation action.

Example: $G = S_3 = \{1, r, r^2, s, sr, sr^2\}$ with $r^3 = 1$, $s^2 = 1$, $rs = sr^{-1}$.

$r \triangleright 1 = r1r^{-1} = 1$

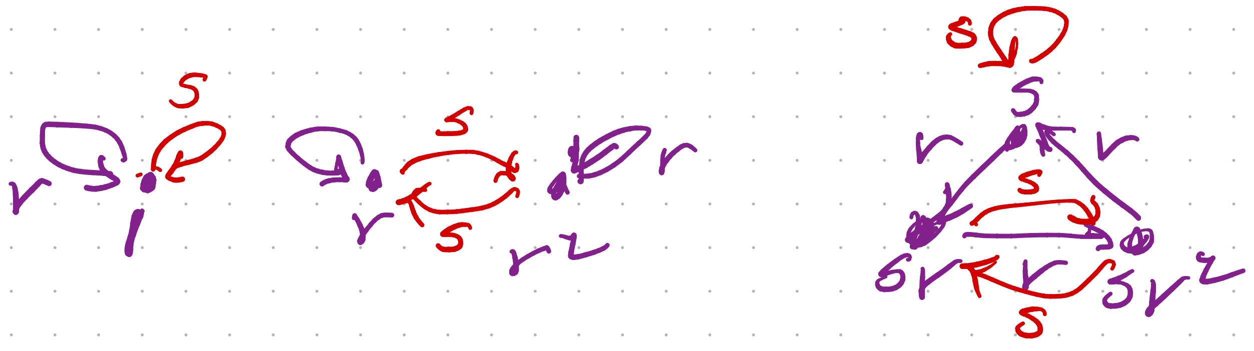
$r \triangleright r = rrr^{-1} = r$

$r \triangleright r^2 = rr^2r^{-1} = r^2$

$r \triangleright s = rsr^{-1} = sr^{-1}r^{-1} = sr^{-2} = sr$

$r \triangleright sr = r(sr)r^{-1} = rs = sr^{-1} = sr^2$

$r \triangleright sr^2 = r(sr^2)r^{-1} = rsr = sr^{-1}r = s.$



$s \triangleright 1 = s1s^{-1} = 1$

$s \triangleright r = srs^{-1} = srs = ssr^{-1} = r^{-1} = r^2$

$s \triangleright r^2 = sr^2s^{-1} = sr^2s = ssr^{-2} = r^{-2} = r$

$s \triangleright s = sss^{-1} = s$

$s \triangleright sr = ssrs^{-1} = rs = sr^2$

$s \diamond sr^2 = s(sr^2)s^{-1} = r^2 s = sr^{-2} = sr$

So

6 $Z_G(1) = \{1, r, r^2, s, sr, sr^2\} = G$.

3 $Z_G(r) = \{1, r, r^2\}$

3 $Z_G(r^2) = \{1, r, r^2\}$.

2 $Z_G(s) = \{1, s\}$

2 $Z_G(sr) = \{1, sr\}$

2 $Z_G(sr^2) = \{1, sr^2\}$.

These are all groups.

$Z_G(g \diamond x) = g Z_G(x) g^{-1}$

1 $\mathcal{C}_1 = \{1\}$

2 $\mathcal{C}_r = \{r, r^2\}$

2 $\mathcal{C}_{r^2} = \{r, r^2\}$

3 $\mathcal{C}_s = \{s, sr, sr^2\} = \underset{3}{\mathcal{C}_{sr}} = \underset{3}{\mathcal{C}_{sr^2}}$.

So S_3 has conjugacy classes $\{1\}$, $\{r, r^2\}$ and $\{s, sr, sr^2\}$

$\mathrm{Card}(Z_G(x))\,\mathrm{Card}(\mathcal{C}_x) = \mathrm{Card}(G)$

$r^2 s \diamond (sr)$

$= r \diamond r \diamond (s \diamond (sr)) = r \diamond (r \diamond sr^2)$

$= r \diamond s = sr$.

So

$r^2 s \in \mathrm{Stab}_G(sr)$.

But $r^2 s = r^{-1} s = sr$.

So $sr \in \mathrm{Stab}_G(sr)$.

Let G be a group.

The <u>center of G</u> is

$$Z(G) = \{ z \in G \mid \text{if } g \in G \text{ then } gz = zg \}$$

$$= \{ z \in G \mid \text{if } g \in G \text{ then } gzg^{-1} = z \}$$

$$= \{ z \in G \mid Z_G(z) = G \}.$$

$$= \{ z \in G \mid G \triangleright z = \{z\} \}.$$

<u>Proposition</u> $Z(G)$ is a normal subgroup of G

<u>Proof</u> To show:

(a) If $z_1, z_2 \in Z(G)$ then $z_1 z_2 \in Z(G)$.

(b) $1 \in Z(G)$

(c) If $z \in Z(G)$ then $z^{-1} \in Z(G)$.

(d) If $z \in Z(G)$ and $g \in G$ then
$$gzg^{-1} \in Z(G).$$

(a) Assume $z_1, z_2 \in Z(G)$

To show: $z_1 z_2 \in Z(G)$.

To show: If $g \in G$ then
$$g(z_1 z_2) = (z_1 z_2)g$$

Assume $g \in G$.

$$g(z_1 z_2) = g z_1 z_2 = z_1 g z_2 \quad \left(\begin{array}{l}\text{since}\\ z_1 \in Z(G)\end{array}\right)$$
$$= z_1 z_2 g \quad \left(\begin{array}{l}\text{since}\\ z_2 \in Z(G)\end{array}\right)$$
$$= (z_1 z_2) g.$$

So $z_1 z_2 \in Z(G)$.

(b) To show: $1 \in Z(G)$.

Assume $g \in G$.

Then $g \cdot 1 = g = 1 \cdot g$.

So $1 \in Z(G)$.

(c) Assume $z \in Z(G)$

To show: $z^{-1} \in Z(G)$.

To show: If $g \in G$ then $gz^{-1} = z^{-1}g$.

Assume $g \in G$.

~~Then~~ $zg = gz$, since $z \in Z(G)$.

Multiply on the left and right by z^{-1} to get

$$gz^{-1} = z^{-1}g.$$

So $z^{-1} \in Z(G)$

(d) To show: If $z \in Z(G)$ and $g \in G$ then $gzg^{-1} \in Z(G)$.

Assume $z \in Z(G)$ and $g \in G$.

To show: $gzg^{-1} \in Z(G)$.

$$gzg^{-1} = zgg^{-1} \quad (\text{since } z \in Z(G))$$
$$= z \cdot 1 = z \in Z(G).$$

So $Z(G)$ is a <u>normal</u> <u>subgroup</u> of G.

$$Z(G) = \{ z \in G \mid \text{If } g \in G \text{ then } gz = zg \}$$

$$= \{ z \in G \mid G \cdot z = \{z\} \}.$$

So $Z(G)$ is the union of the orbits under conjugation action (the conjugacy classes of G) which size 1.

If S is a G-set then the orbits partition S.

So

$$\text{Card}(S) = \sum_{\text{distinct orbits.}} \text{Card}(G \cdot x_i)$$

For the conjugation action

$$\text{Card}(G) = \sum_{\text{distinct conj. classes}} \text{Card}(\mathcal{C}_{x_i}).$$

In our example of $G = S_3$, this

is $6 = 1 + 2 + 3$.

The Class equation

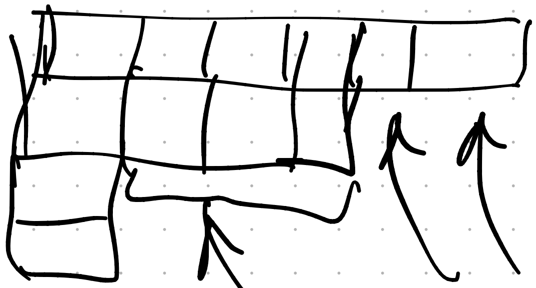
Let G be a group.

$$\mathrm{Card}(G) = \mathrm{Card}(Z(G)) + \sum_{\substack{\text{conj. classes} \\ \text{with} \\ \mathrm{Card}(\mathcal{C}_{x_i}) > 1}} \mathrm{Card}(\mathcal{C}_{x_i}).$$

This is understood since $Z(G)$ is the union of the conjugacy classes size 1 and G is the union of all the conjugacy classes.

This completes everything needed for questions 3, 4, 5, 6 on Ass. 3.

$$\boxed{n! = \sum_{\substack{\text{pertitons}\\ \text{of } n.}} 1^{m_1} 2^{m_2} \cdots \; m_1! \, m_2! \cdots}$$



n boxes shoved in a corner.

$m_1 = \#$ columns of length 1.

$m_2 = \#$ colums of length 2

$$e^x e^y = e^{x+y}.$$

$$\boxed{\text{multizeta functions}}$$