GTLA Lecture 25.04.2020

Let $\mathbb{F}$ be a field.

Let V be an $\mathbb{F}$-vectorspace.

Let $\langle , \rangle : V \times V \to \mathbb{F}$ be a sesquilinear form.

Let W be a subspace of V.

Let $k \in \mathbb{Z}_{>0}$ such that $\dim(W) = k$.

Let's fix a basis $B = \{w_1, \dots, w_k\}$ of W.

The <u>Gram matrix</u> of $\langle , \rangle$ with <u>respect to B</u> is $G \in M_k(\mathbb{F})$

$$G(i;j) = \langle w_i ; w_j \rangle.$$

The <u>dual basis to</u> $\{w_1, \dots, w_k\}$ is $\{w^1, \dots, w^k\}$ such that

$$\langle w^i ; w_j \rangle = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

<u>Example</u> $V = \mathbb{R}^3$ with the standard dot product.

Let $W = \text{span}\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\0\\4\end{pmatrix} \right\}$.

Fix basis $\{w_1, w_2\}$ with

$$w_1 = \begin{pmatrix}1\\1\\1\end{pmatrix} \text{ and } w_2 = \begin{pmatrix}2\\0\\4\end{pmatrix}.$$

The Gram matrix of dot product with respect to $\{w_1, w_2\}$ is

$$G = \begin{pmatrix}3 & 6\\ 6 & 20\end{pmatrix} \text{ since}$$

$$w_1 \cdot w_1 = \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = 3, \quad \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}2\\0\\4\end{pmatrix} = 6$$

$$\begin{pmatrix}2\\0\\4\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = 6, \quad \begin{pmatrix}2\\0\\4\end{pmatrix} \cdot \begin{pmatrix}2\\0\\4\end{pmatrix} = 4+16 = 20$$

The dual basis to $\{w_1, w_2\}$ is $\{w^1, w^2\}$ given by

$$w^1 = \begin{pmatrix}\frac{1}{3}\\ \frac{5}{6}\\ -\frac{1}{6}\end{pmatrix} \text{ and } w^2 = \begin{pmatrix}0\\ -\frac{1}{4}\\ \frac{1}{4}\end{pmatrix}$$

CHECK:

$$w^1 \cdot w_1 = \begin{pmatrix} \frac{1}{3} \\ \frac{5}{6} \\ -\frac{1}{6} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3} + \frac{5}{6} - \frac{1}{6} = \frac{2}{6} + \frac{5}{6} - \frac{1}{6}$$

$$= \frac{6}{6} = 1$$

$$w^1 \cdot w_2 = \begin{pmatrix} \frac{1}{3} \\ \frac{5}{6} \\ -\frac{1}{6} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = \frac{2}{3} - \frac{4}{6} = 0$$

$$w^2 \cdot w_1 = \begin{pmatrix} 0 \\ -\frac{1}{4} \\ \frac{1}{4} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 - \frac{1}{4} + \frac{1}{4} = 0$$

$$w^2 \cdot w_2 = \begin{pmatrix} 0 \\ -\frac{1}{4} \\ \frac{1}{4} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = 0 + 0 + \frac{4}{4} = 1.$$

$$w^i \cdot w_j = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

$$w^1 = \begin{pmatrix} \frac{1}{3} \\ \frac{5}{6} \\ -\frac{1}{6} \end{pmatrix} = \frac{5}{6} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{-1}{4} \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = \frac{5}{6} w_1 - \frac{1}{4} w_2$$

$$w^2 = \begin{pmatrix} 0 \\ -\frac{1}{4} \\ \frac{1}{4} \end{pmatrix} = \frac{-1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{8} \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = \frac{-1}{4} w_1 + \frac{1}{8} w_2$$

is

$$G^{-1} = \begin{pmatrix} \frac{5}{6} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{8} \end{pmatrix}$$ since

$$\begin{pmatrix} \frac{5}{6} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{8} \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 6 & 20 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is an example of

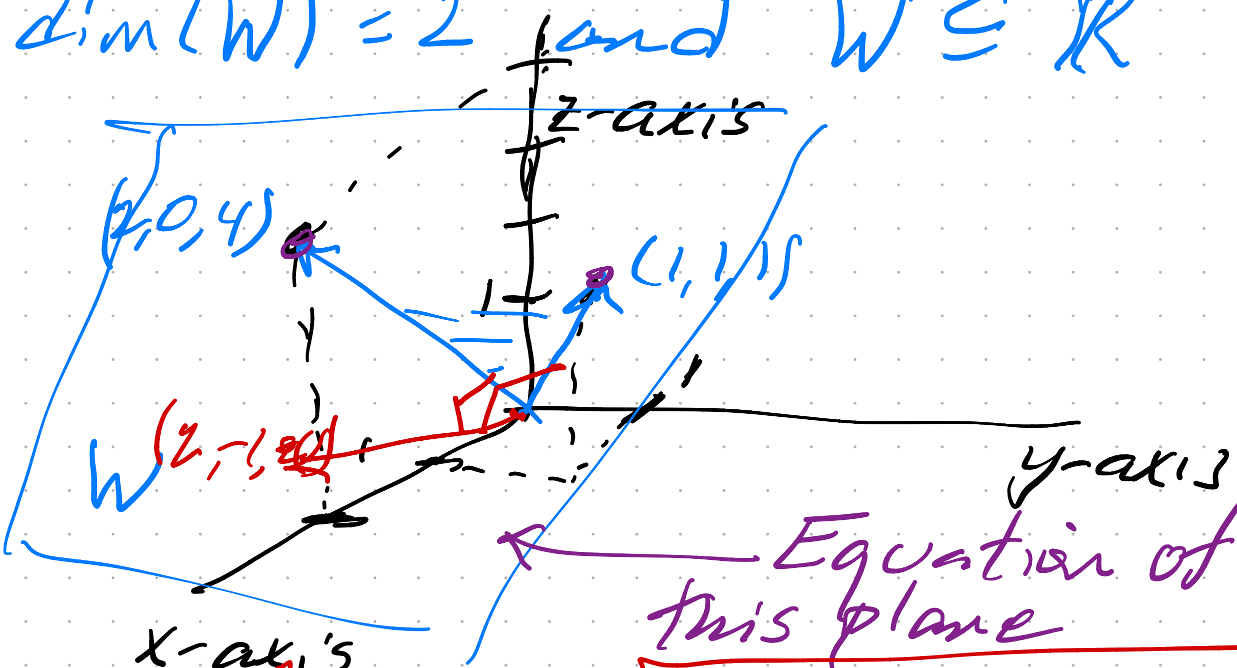
$$w^i = \sum_{l=1}^{k} G^{-1}(l,i)\, w_l .$$

from last time.

Staying with the same example

$$W = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \right\}$$

$\dim(W) = 2$ and $W \subseteq \mathbb{R}^3$



Equation of this plane

$$\boxed{2x - y - z \overset{?}{=} 0}$$

$2\cdot 1 - 1 - 1 \overset{?}{=} 0$

$2\cdot 2 - 0 - 4 \overset{?}{=} 0$

$(1,1,1) \in W$ and $(2,0,4) \in W$.

$$\boxed{W^\perp = \{ v \in V \mid \text{if } w \in W \text{ then } \langle v, w \rangle = 0 \}.}$$

$$W^\perp = \text{span}\left\{ \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right\}$$

orthogonal defined always

$\dim(W) = 2$, $\dim(W^\perp) = 1$

$W \leq \mathbb{R}^3$ and $W^\perp \leq \mathbb{R}^3$

We're heading towards a theorem which says if $W \cap W^\perp = 0$ then $V = W \oplus W^\perp$

$3 = 2 + 1.$

Orthogonal projection onto W

Let $\langle , \rangle : V \times V \to \mathbb{F}$ be a sesquilinear form. Let $W \leq V$ a subspace. Let

$\{w_1, \dots, w_k\}$ be a basis of W.

Assume $W \cap W^\perp = 0$ so that the dual basis exists $\{w^1, w^2, \dots, w^k\}$.

The <u>orthogonal projection onto W</u> is the linear transformation $P_W : V \to V$ given

$$P_W(v) = \sum_{i=1}^{k} \langle v, w_i \rangle w^i$$

defined only if $W \cap W^\perp = 0$

$$= \langle v, w_1 \rangle w^1 + \langle v, w_2 \rangle w^2 + \cdots$$
$$\cdots + \langle v, w_k \rangle w^k$$

If $v \in V$ then $P_W(v) \in W$.

(Does P_W depend on the choice of the basis?)

<u>Proposition</u> P_W is the unique linear transformation $P : V \to V$ such that

(1) If $v \in V$ then $P(v) \in W$

(2) If $v \in V$ and $w \in W$ then
$$\langle v, w \rangle = \langle P(v), w \rangle.$$

Proof To show:

(a) P_W satisfies (1) and (2).

(b) If $P: V \to V$ and $Q: V \to V$ both satisfy (1) and (2) then $P = Q$.

(a) We already noticed P_W satisfies (1).

To show: If $v \in V$ and $w \in W$ then

$$\langle v, w \rangle = \langle P_W(v), w \rangle.$$

Assume $v \in V$ and $w \in W$.

Write $w = c_1 w_1 + \cdots + c_k w_k$.

$$\langle P_W(v), w \rangle = \left\langle \sum_{i=1}^{k} \langle v, w_i \rangle w^i, w \right\rangle$$

$$= \sum_{i=1}^{k} \langle v, w_i \rangle \langle w^i, w \rangle$$

$$= \sum_{i=1}^{k} \langle v, w_i \rangle \langle w^i, c_1 w_1 + \cdots + c_k w_k \rangle$$

$$= \sum_{i=1}^{k} \langle v, w_i \rangle \left(\bar{c}_1 \langle w^i, w_1 \rangle + \cdots + \bar{c}_k \langle w^i, w_k \rangle \right)$$

$$= \sum_{i=1}^{k} \langle v, w_i \rangle \left(\bar{c}_1 \cdot 0 + \cdots + \bar{c}_i \cdot 1 + \bar{c}_{i+1} \cdot 0 + \cdots + \bar{c}_k \cdot 0 \right)$$

$$= \sum_{i=1}^{k} \langle v, w_i \rangle \bar{c}_i = \sum_{i=1}^{k} \bar{c}_i \langle v, w_i \rangle$$

$$= \sum_{i=1}^{k} \langle v, c_i w_i \rangle = \langle v, w \rangle.$$

So P_W satisfies property (2).

(b) If P, Q both satisfy (1) and (2) then $P = Q$.

Assume P, Q both satisfy (1) and (2)

To show: If $v \in V$ then $P(v) = Q(v)$

Assume $v \in V$.

By property (1),

$$P(v) - Q(v) \in W$$

By property (2), if $w \in W$ then

$$\langle P(v) - Q(v), w \rangle$$

$$= \langle P(v), w \rangle - \langle Q(v), w \rangle$$

$$= \langle v, w \rangle - \langle v, w \rangle = 0.$$

So $P(v) - Q(v) \in W^\perp$.

So $P(v) - Q(v) \in W \cap W^\perp$.

Since $W \cap W^\perp = 0$ then

$$P(v) - Q(v) = 0.$$

So $P(v) = Q(v)$ and $P = Q$. //