GTLA Lecture 03.09.2020

A **group** is a set G with a function
$$G \times G \to G \qquad \text{such that}$$
$$(a, b) \mapsto a \heartsuit b$$

(a) If $g_1, g_2, g_3 \in G$ then
$$g_1 \heartsuit (g_2 \heartsuit g_3) = (g_1 \heartsuit g_2) \heartsuit g_3.$$

(b) there exists ☺ $\in G$ such that if $g \in G$ then ☺ $\heartsuit g = g$ and $g \heartsuit$ ☺ $= g$.

(c) If $g \in G$ then there exists ß $\in G$ such that
$g \heartsuit$ ß $=$ ☺ and ß $\heartsuit g =$ ☺

A **subgroup of G** is a subset $H \subseteq G$ such that

(a) If $h_1, h_2 \in H$ then $h_1 \heartsuit h_2 \in H$

(b) ☺ $\in H$

(c) If $h \in H$ then $4 \in H$.

A group G is <u>commutative</u>, or <u>abelian</u>, if G satisfies:

if $g_1, g_2 \in G$ then $g_1 \heartsuit g_2 = g_2 \heartsuit g_1$.

Homomorphisms are for comparing groups.

Let H and K be groups.

A <u>homomorphism</u> <u>from H to K</u> is a function $f: H \to K$ such that

(a) If $h_1, h_2 \in H$ then
$$f(h_1 \heartsuit h_2) = f(h_1) f(h_2).$$

(b) $f(\boxed{☺}) = 1$.

(c) If $h \in H$ then $f(h^{-1}) = f(h)^{-1}$.

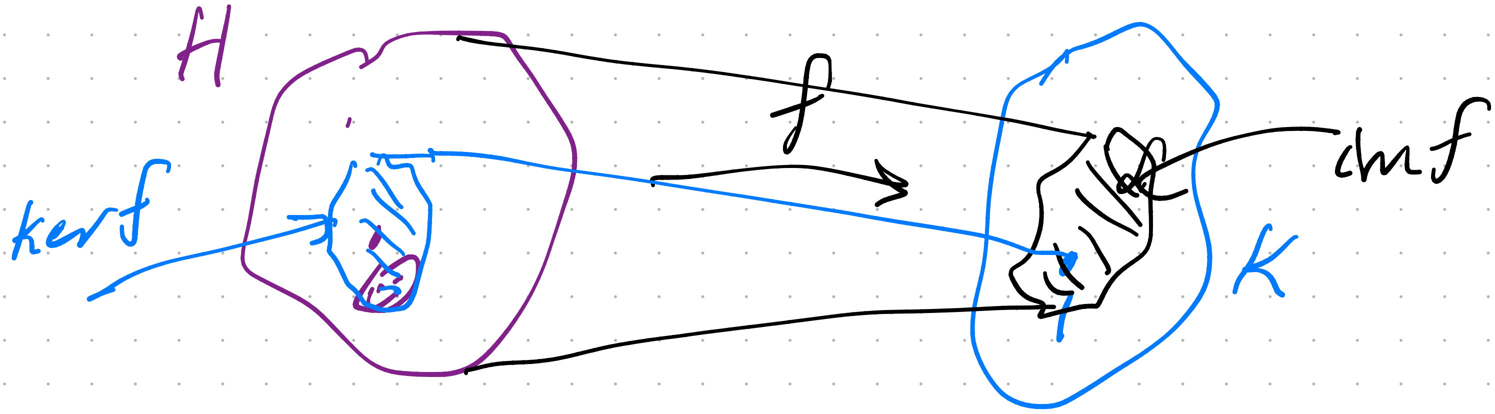
An <u>isomorphism</u> from H to K is a homomorphism from H to K which is bijective.

Let $f: H \to K$ be a homomorphism.

The <u>kernel</u> of f is
$$\ker(f) = \{h \in H \mid f(h) = 1\}.$$

The image of f is

$$\operatorname{im}(f) = \{ f(h) \mid h \in H \}.$$



Proposition (a) $\ker f$ is a subgroup of H

(b) $\operatorname{im} f$ is a subgroup of K

Examples of Groups

(1) $GL_n(\mathbb{F}) = \left\{ \begin{array}{c} \text{invertible matrices} \\ \text{in } M_n(\mathbb{F}) \end{array} \right\}$

(2) $SL_n(\mathbb{F}) = \left\{ \begin{array}{c} \text{matrices with} \\ \text{determinant } 1 \text{ in } M_n(\mathbb{F}) \end{array} \right\}$.

⊛ (3) S_n = symmetric group.

⊛ (4) C_n = cyclic groups

⊛ (5) D_n = dihedral groups.

$$C_1 = \{(1)\} \qquad \mathrm{Card}(C_1) = 1.$$

$$C_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \qquad \mathrm{Card}(C_2) = 2.$$

$$C_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\}$$

$$\mathrm{Card}(C_3) = 3.$$

$$C_4 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\}$$

$$\mathrm{Card}(C_4) = 4$$

$$D_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

$\mathrm{Card}(D_3) = 6.$

$$D_4 = \left\{ \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0 \end{pmatrix} \right\}$$

$\mathrm{Card}(D_4) = 8.$

The group $\mathbb{C}^\times$

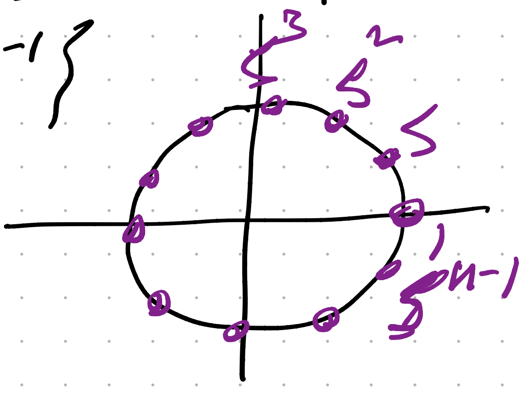
$$\mathbb{C}^\times = GL_1(\mathbb{C}) = \{(c) \mid c \text{ is invertible in } \mathbb{C}\}$$
$$= \mathbb{C} - \{0\}. \text{ under multiplication.}$$

$\mathrm{Card}(\mathbb{C}^\times) = \infty.$

The group μ_n

$$\mu_n = \{n^{\text{th}} \text{ roots of } 1 \text{ in } \mathbb{C}\}$$
$$= \{e^0, e^{2\pi i/n}, e^{2\pi i\, 2/n}, \ldots, e^{2\pi i \frac{(n-1)}{n}}\}$$
$$= \{1, \zeta, \zeta^2, \ldots, \zeta^{n-1}\}$$

where $\zeta = e^{2\pi i/n}.$



$\mathrm{Card}(\mu_n) = n.$

Then $\mu_n \cong \mathbb{Z}/n\mathbb{Z} \cong C_n$. (more on this on our todo list).

Let G be a group.

(The order of G is $\mathrm{Card}(G)$).

Let $g \in G$.

The <u>order of g</u> is the smallest $k \in \mathbb{Z}_{>0}$ such that $g^k = 1$.

Example

If $G = \mathbb{C}^\times$ and $g = 2$.

$\mathrm{order}(g) = \mathrm{order}(2) = \infty$.

<u>Different notations for elements of S_n</u>

In S_6,

two line notation.

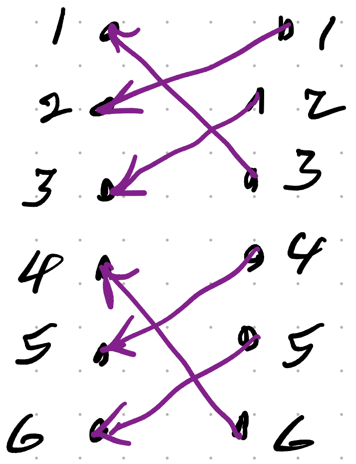
$$w = \begin{pmatrix} 0&0&1&0&0&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \end{pmatrix} = \begin{pmatrix} 1&2&3&4&5&6 \\ 2&3&1&5&6&4 \end{pmatrix}$$

Let $e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ith

$we_1 = e_2$, $\quad we_4 = e_5$

$we_2 = e_3$, $\quad we_5 = e_6$

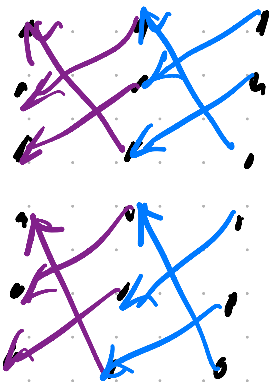
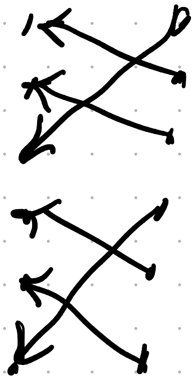
$we_3 = e_1$, $\quad we_6 = e_4$

$w = (231564) =$  function notation.

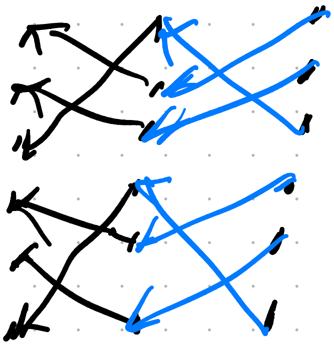
one line notation is bottom line of two line notation

$= (123)(456)$ cycle notation

Find order(w).

$w^2 = ww =$  $=$ 

$w \cdot w$

$w^3 = w^2 \cdot w =$  $=$  $= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$ identity

$w^2 \cdot w$

So $w^3 = 1$. So order$(w) = 3$

Isomorphism:

$\zeta = e^{2\pi i/n}$

$$\begin{array}{ccc}
\mu_n & \longrightarrow & \mathbb{Z}/n\mathbb{Z} \\
1 & \longmapsto & 0 \\
\zeta & \longmapsto & 1 \\
\zeta^2 & \longmapsto & 2 \\
\vdots & & \vdots \\
\zeta^{n-1} & \longmapsto & n-1
\end{array}$$

$\mathrm{Card}(\mathbb{C}^\times) = \infty$.

$\mathrm{order}(2) = \infty$

$\mathrm{order}(-1) = 2$.

$\mathrm{order}(e^{2\pi i/n}) = n$

$(-1)^2 = 1$.

$2, e^{2\pi i/n}, -1 \in \mathbb{C}^\times$.