GTLA Lecture 04.09.2020

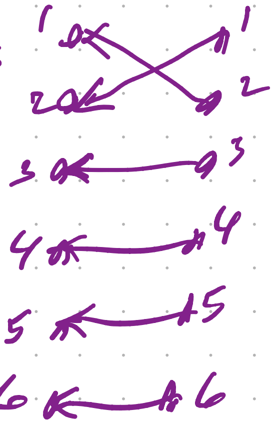
The symmetric group S_6

matrix notation
two line notation
one line notation

function notation
cycle notation.

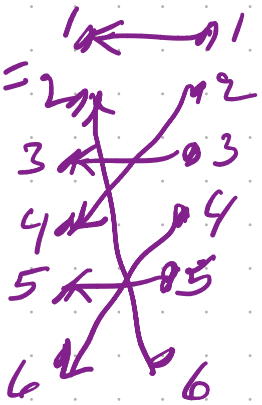
Compute $(12) \times (246) \times (123654)$

$w_1 = (12) =$



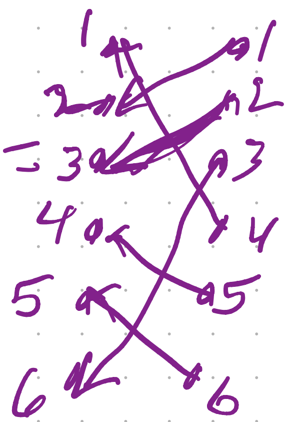
$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$w_2 = (246) =$



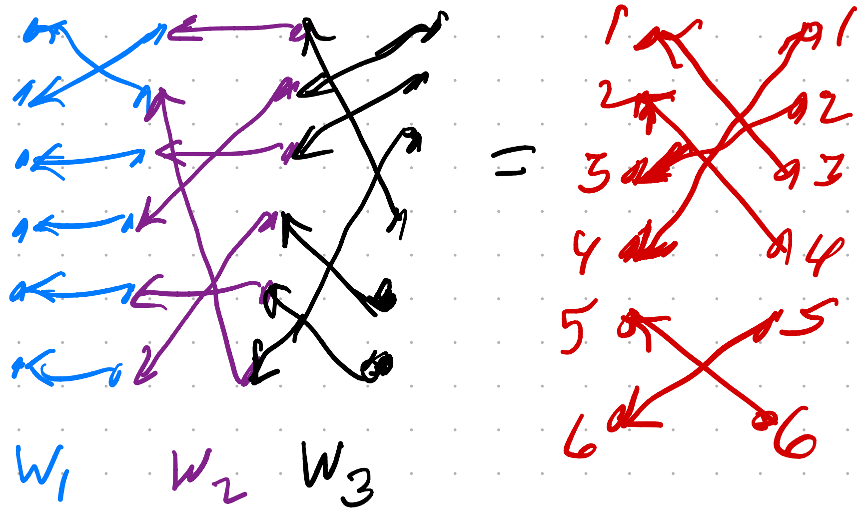
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$w_3 = (123654) =$



$$= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$W_1 W_2 W_3 =$

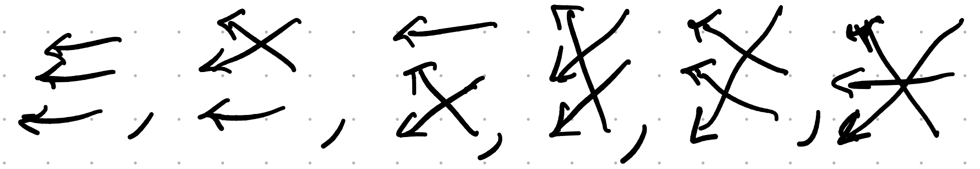


$= (1423)(56)$ in cycle notation

$= (4231)(65)$

$= (2314)(56)$

$$S_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}$$

$=$ {  }

$= \{ 1, (12), (23), (123), (132), (13) \}$

Let G be a group.

Let S be a subset of G.

The <u>subgroup</u> of G <u>generated by S</u> is the subgroup H such that

(a) $H \supseteq S$.

(b) If K is a subgroup and $K \supseteq S$ then $K \supseteq H$.

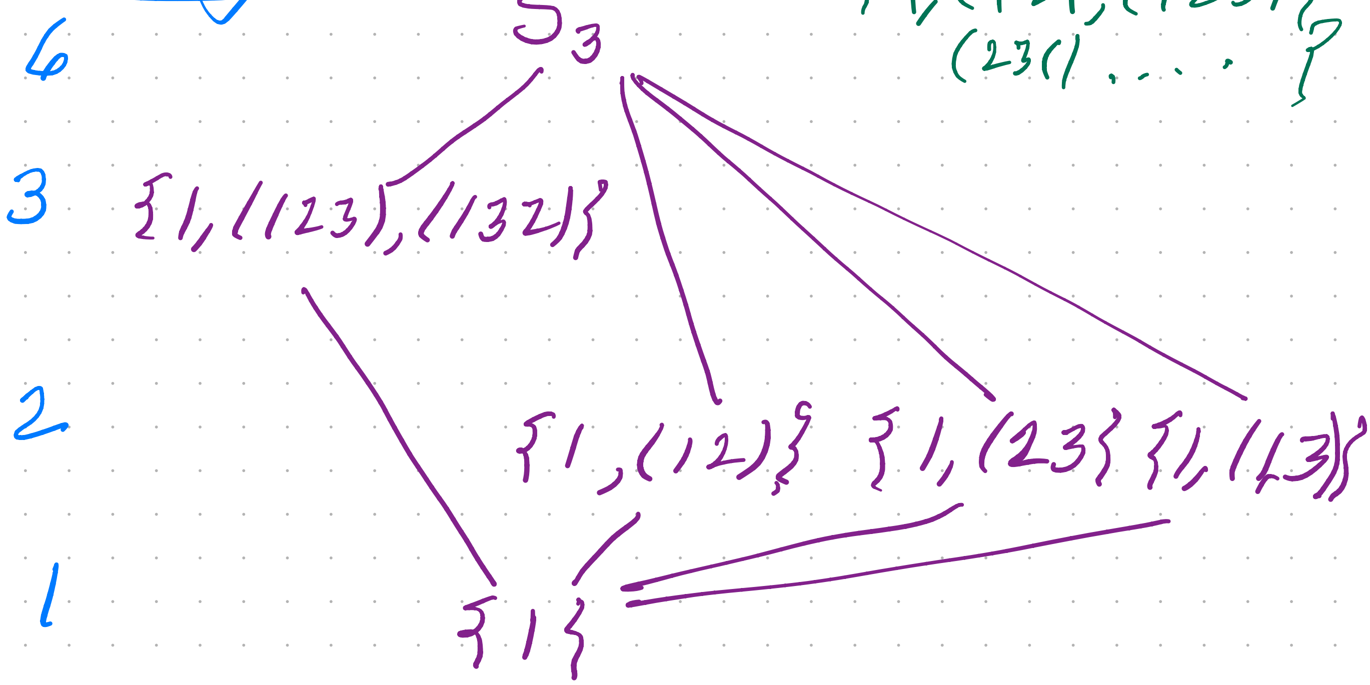
In English: H is the smallest subgroup containing S.

Note the similarity in structure to one of characterisations of gcd (or lcm): this is a "universal property".

Subgroups of S_3

Cardinality

$\{1, (12), (123), (231) \ldots \}$



Let $S = \{(12), (13)\}$ subset of S_3

Let H be ~~So~~ the subgroup generated by 5.

So $H = S_3$

Let G be a group and $g \in G$.
The <u>order of g</u> is the smallest $k \in \mathbb{Z}_{>0}$ such that

$$\underbrace{g \circledast g \circledast g \circledast \cdots \circledast g}_{k \text{ times}} = ☺$$

If $k \in \mathbb{Z}_{>0}$ doesn't exist then $\text{order}(g) = \infty$.

<u>The group $\mathbb{Z}/10\mathbb{Z}$</u>

$\mathbb{Z}/10\mathbb{Z} = \{0,1,2,3,4,5,6,7,8,9\}$

$9+9+9+9+9+9+9+9+9+9 = 0$
So $\text{order}(9) = 10$.

$8+8+8+8+8 = 0$ So $\text{order}(8) = 5$.

$7+7+7+7+7+7+7+7+7+7 = 0$
So $\text{order}(7) = 10$

$6+6+6+6+6 = 0$ So $\text{order}(6) = 5$.

$5+5 = 0$ SO $\text{order}(5) = 2$.

$4+4+4+4+4 = 0$ So $\text{order}(4) = 5$

$3+3+3+3+3+3+3+3+3+3=0$

So order$(3)=10$

$2+2+2+2+2=0$ So order$(2)=5$

$1+1+1+1+1+1+1+1+1+1=0$

So order$(1)=10$

$0=0$ So order$(0)=1$.

Subgroups of $\mathbb{Z}/10\mathbb{Z}$

Warmup: Subgroup generated by 3.

$\{0,3,6,9,2,5,8,1,4,7\}$

$=\{0,1,2,3,4,5,6,7,8,9\}=\mathbb{Z}/10\mathbb{Z}$.

Cardinality

$$\begin{tikzcd}
10 & & \mathbb{Z}/10\mathbb{Z} \arrow[dl, dash] \arrow[ddr, dash] & \\
5 & \{0,2,4,6,8\} \arrow[ddr, dash] & & \\
2 & & & \{0,5\} \arrow[dl, dash] \\
1 & & \{0\} &
\end{tikzcd}$$

Proposition Let G be group and let $g \in G$.

(a) Let $k \in \mathbb{Z}_{>0}$ and assume $\mathrm{order}(g) = k$. Then the subgroup generated by g is

$$\{1, g, g^2, g^3, \ldots, g^{k-1}\} \text{ and}$$

$$\mathrm{Card}\{1, g, g^2, \ldots, g^{k-1}\} = \mathrm{order}(g).$$

(b) Assume $\mathrm{order}(g) = \infty$. Then the subgroup generated by g

$$\{\ldots g^{-3}, g^{-2}, g^{-1}, 1, g, g^2, g^3, \ldots \ldots\} = \langle g \rangle$$

and

$$\mathrm{Card}(\langle g \rangle) = \infty.$$

Let G be a group. A <u>cyclic subgroup of G</u> is a subgroup generated by one element.

$(1\,6\,2\,3\,5\,7\,4)$ in cycle notation

$(1\,6\,2\,3\,5\,7\,4)$ in one-line notation.

$(G, \heartsuit)$ $\qquad (G, \cdot)$