GTLA Lecture 20.08.2020

Let $\mathbb{F}$ be a field.
Let $A \in M_n(\mathbb{F})$.

The <u>minimal polynomial of A</u> $m_A(x)$ is the smallest degree monic (top coeff. is 1) polynomial such that

$$m_A(A) = 0.$$

The <u>characteristic polynomial</u> of A is $\det(x-A)$

<u>Theorem</u> (Cayley-Hamilton)
$\det(x-A)$ is a multiple of $m_A(x)$.

<u>Example</u> $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\det(x-A) = \det\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} x-1 & 0 \\ 0 & x-1 \end{pmatrix} = (x-1)^2.$$

Plug in A:

$$(A-1)^2 = \left(\begin{pmatrix}1&0\\0&1\end{pmatrix} - \begin{pmatrix}1&0\\0&1\end{pmatrix}\right)^2$$

$$= \begin{pmatrix}0&0\\0&0\end{pmatrix}^2 = \begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

$m_A(x) = x-1.$

$$m_A(A) = A-1 = \begin{pmatrix}1&0\\0&1\end{pmatrix} - \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$= \begin{pmatrix}0&0\\0&0\end{pmatrix} = 0.$$

$$\det(x-A) = (x-1)^2 = x^2-2x+1$$

$m_A(x) = x-1.$

Let $A \in M_n(\mathbb{F})$.

The matrix A is <u>diagonalisable</u> over $\mathbb{F}$ if there exist $P \in GL_n(\mathbb{F})$ ($GL_n(\mathbb{F})$ is invertible matrices) and $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ such that

$$P^{-1}AP = \begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$$

Write
$$\mathrm{diag}(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Let $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.

$\det(D) = \lambda_1 \lambda_2 \cdots \lambda_n$

If A is diagonalisable then

$$\begin{aligned}\det(A) &= \det(P)^{-1}\det(P)\det(A) \\ &= \det(P)^{-1}\det(A)\det(P) \\ &= \det(P^{-1})\det(A)\det(P) \\ &= \det(P^{-1}AP) \\ &= \det(D) = \lambda_1 \cdots \lambda_n.\end{aligned}$$

$$\det(x-D) = \det\left(\begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} - \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} x-\lambda_1 & & 0 \\ & \ddots & \\ 0 & & x-\lambda_n \end{pmatrix}$$

$= (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$

In this way we know, or calculate the characteristic polynomial of diagonalisable matrix.

How ~~do~~ do we know if A is diagonalisable.

<u>Theorem</u> Let $A \in M_n(\mathbb{F})$. Then A is diagonalisable if and only if there exist n linearly independent eigenvectors of A

<u>Proof</u> (Sketch). $\Longrightarrow$

Assume $P^{-1}AP = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Then

$$AP = P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Let $p_1 = \begin{pmatrix} p_{11} \\ \vdots \\ p_{n1} \end{pmatrix}, p_2 = \begin{pmatrix} p_{12} \\ \vdots \\ p_{n2} \end{pmatrix} \cdots p_n = \begin{pmatrix} p_{1n} \\ \vdots \\ p_{nn} \end{pmatrix}$

$$A \begin{pmatrix} | & | & & | \\ p_1 & p_2 & \cdots & p_n \\ | & | & & | \end{pmatrix} = A \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{1n} & \cdots & p_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{1n} & & p_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$= \begin{pmatrix} p_{11}\lambda_1 & \cdots & p_{1n}\lambda_n \\ \vdots & \cdots & \vdots \\ p_{1n}\lambda_1 & \cdots & p_{nn}\lambda_n \end{pmatrix}$$

$$= \begin{pmatrix} | & & | \\ \lambda_1 p_1 & \cdots & \lambda_n p_n \\ | & & | \end{pmatrix}.$$

So $A p_j = \lambda_j p_j$. [The columns of P are the eigenvectors].

Use a theorem from before

$\{\text{bases of } \mathbb{F}^n\} \longleftrightarrow GL_n(\mathbb{F})$

$(p_1, \ldots, p_n) \longleftrightarrow \begin{pmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{pmatrix} = P.$

So P being invertible implies the columns are linearly independent.

This completes the proof of $\Rightarrow$. //.

<u>Not diagonizable over $\mathbb{R}$ but which is diagonalisable over $\mathbb{C}$.</u>

Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then

$$\det(x - A) = \det\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} x & -1 \\ 1 & x \end{pmatrix} = x^2 + 1.$$

There is no real number $\lambda \in \mathbb{R}$ such that $\lambda^2 + 1 = 0$.

So $\det(\lambda - A) \neq 0$ if $\lambda \in \mathbb{R}$.

So $\ker(\lambda - A) = 0$ if $\lambda \in \mathbb{R}$.

So $A \in M_2(\mathbb{R})$ has no eigenvectors in $\mathbb{R}^2$.

So A is not diagonalisable over $\mathbb{R}$.

$\det(x-A) = x^2+1 = (x-i)(x+i)$

so i and $-i$ are roots of x^2+1.

i.e. $i^2+1=0$ and $(-i)^2+1=0$

So over $\mathbb{C}$, there is an eigenvector p_1 of eigenvalue i,

and another eigenvector p_2 of eigenvalue $-i$.

Since i and $-i$ are distinct then p_1 and p_2 are linearly independent.

So if $P = \begin{pmatrix} | & | \\ p_1 & p_2 \\ | & | \end{pmatrix}$ then

$$P^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}P = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

To find p_1 we want $Ap_1 = ip_1$

(can multiply p_1 by any nonzero constant and still $A(\mu p_1) = i(\mu p_1)$)

So assume $p_1 = \begin{pmatrix} 1 \\ c \end{pmatrix}$.

Want
$$Ap_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ c \end{pmatrix} = i\begin{pmatrix} 1 \\ c \end{pmatrix} = ip_1.$$

So
$$\begin{pmatrix} c \\ -1 \end{pmatrix} = i\begin{pmatrix} 1 \\ c \end{pmatrix}.$$

So $c = i$. So $p_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

To find p_2, assume $p_2 = \begin{pmatrix} 1 \\ d \end{pmatrix}$.

Want
$$Ap_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ d \end{pmatrix} = (-i)\begin{pmatrix} 1 \\ d \end{pmatrix}$$
$$= -ip_2.$$

So
$$\begin{pmatrix} d \\ -1 \end{pmatrix} = \begin{pmatrix} -i \\ -id \end{pmatrix}$$

So $d = -i$. So $p_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

So $P = \begin{pmatrix} | & | \\ p_1 & p_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$.

and
$$P^{-1}AP = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

So A is diagonalisable over $\mathbb{C}$.

If $A \in M_n(\mathbb{F})$ is diagonalisable over $\mathbb{F}$ then A has n linearly independent eigenvectors in $\mathbb{F}^n$.

<u>2x2 matrix in $M_2(\mathbb{C})$ with only</u>
<u>**one** lin. indep. eigenvector in $\mathbb{C}^2$.</u>

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{C})$.

Then $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

So $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of eigenvalue 1.

$$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is <u>not</u> an eigenvector.

If $A\begin{pmatrix} 1 \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ c \end{pmatrix}$ then

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ c \end{pmatrix}$. So $\begin{pmatrix} 1+c \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ c \end{pmatrix}$.

So $1+c=1$. So $c=0$.

So **only** option for an eigenvector of eigenvalue 1 is $p_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\det(x-A) = \det\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} x-1 & -1 \\ 0 & x-1 \end{pmatrix} = (x-1)^2.$$

So the only possible eigenvalue is 1. If $\lambda \neq 1$ then $\det(\lambda-A) \neq 0$.

Examples

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ not diag. over $\mathbb{R}$ but is over $\mathbb{C}$.

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ one eigenvector only.

If $\det(\lambda - A) = 0$ then $\lambda - A$ is not invertible and $\ker(\lambda - A) \neq 0$.

If $\det(\lambda - A) \neq 0$
then $\lambda - A$ is invertible
and $\ker(\lambda - A) = 0$.

$\ker(f) = 0 \Leftrightarrow f$ is injective.