GTLA Lecture 11.09.2020

Quotients - Cosets

Let G be a group and H a subgroup of G.

The set of cosets is

$$G/H = \{gH \mid g \in G\}$$ (set of sets)

where

$$gH = \{gh \mid h \in H\}$$

Example $S_3 = D_3 = G$

$$= \{1, r, r^2, s, sr, sr^2\}$$

where $r^3 = 1$, $s^2 = 1$ and $rs = sr^{-1}$.

Let $H = \{1, r, r^2\}$ (a subgroup of G)

Then

$1 \cdot H = \{1, r, r^2\}$ $\quad$ $sH = \{s, sr, sr^2\}$

$rH = \{r, r^2, 1\}$ $\quad$ $srH = \{sr, sr^2, s\}$

$r^2H = \{r^2, 1, r\}$ $\quad$ $sr^2H = \{sr^2, s, sr\}$.

$$H = rH = r^2H$$

$sH = srH = sr^2H.$

$gh_1H = gH.$

So $G/H = \{H, sH\}$ and

$\mathrm{Card}(G/H) = 2$ and $\mathrm{Card}(H) = 3$

$6 = 2 \cdot 3.$

<u>Theorem</u> (Lagrange's theorem).
Let G be a group and H a subgroup of G

(a) The cosets partition G.

(b) If $g \in G$ then
$$\mathrm{Card}(gH) = \mathrm{Card}(H).$$

(c) $\mathrm{Card}(G) = \mathrm{Card}(G/H)\,\mathrm{Card}(H).$

<u>Proof</u> (b) Assume $g \in G$

To show: $\mathrm{Card}(gH) = \mathrm{Card}(H)$.

To show: There exists a bijection
$$\varphi : H \longrightarrow gH$$

Let

$$\varphi: H \longrightarrow gH$$
$$h \longmapsto gh$$

and

$$\psi: gH \longrightarrow H$$
$$x \longmapsto g^{-1}x.$$

To show: φ and ψ are inverse functions.

Since

$$(\psi \circ \varphi)(h) = \psi(\varphi(h)) = \psi(gh) = g^{-1}gh = h.$$

$$(\varphi \circ \psi)(x) = \varphi(\psi(x)) = \varphi(g^{-1}x) = gg^{-1}x = x.$$

then φ and ψ are inverse functions

So φ is bijective

So $\text{Card}(H) = \text{Card}(gH)$.

<u>Proof of (a)</u>

To show: The cosets partition G.

To show: (~~ab~~) $\bigcup_{g\in G} gH = G$

(ab) If $g_1, g_2 \in G$ and $g_1H \cap g_2H \neq \emptyset$ then $g_1H = g_2H$.

(aa) To show: (aaa) $\bigcup_{g\in G} gH \subseteq G$

(aab) $G \subseteq \bigcup_{g\in G} gH$.

(aaa) Since (H ⊆ G and G is closed)
$gH = \{gh \mid h\in H\} \subseteq G$
then $\bigcup_{g\in G} gH \subseteq G$.

(aab) If $a\in G$ then there exists $g\in G$ such that $a\in gH$.

Assume $a\in G$

To show: There exists $g\in G$ such that $a\in gH$.

Let $g=a$

Then $a=g=g\cdot 1\in gH$.

So $\boxed{G=\bigcup_{g\in G} gH=\bigcup_{gH\in G/H} gH.}$

(ab) To show: If $g_1, g_2\in G$ and

$g_1H\cap g_2H\neq\phi$ then $g_1H=g_2H$.

Assume $g_1,g_2\in G$ and

$g_1H\cap g_2H\neq\phi$.

Let $z\in g_1H\cap g_2H$.

Then there exists $h_1\in H$ such

that $z=g_1h_1$ and

there exists $h_2\in H$ such that

$z=g_2h_2$.

So $g_1=zh_1^{-1}=g_2h_2h_1^{-1}$

and $g_2=zh_2^{-1}=g_1h_1h_2^{-1}$.

To show: $g_1H=g_2H$.

To show: (aba) $g_1 H \subseteq g_2 H$

(abb) $g_2 H \subseteq g_1 H$.

(aba) Let $x \in g_1 H$

To show: $x \in g_2 H$

Since $x \in g_1 H$ there exists $q \in H$ such that $x = g_1 q$

Then

$$x = g_1 q = g_2 h_2 h_1^{-1} q$$
$$= g_2 (h_2 h_1^{-1} q) \in g_2 H.$$

So $g_1 H \subseteq g_2 H$.

(abb) To show: $g_2 H \subseteq g_1 H$

To show: If $y \in g_2 H$ then $y \in g_1 H$.

Assume $y \in g_2 H$.

Then there exists $p \in H$ such that $y = g_2 p$.

To show: $y \in g_1 H$.

$$y = g_2 p = g_1 h_1 h_2^{-1} p = g_1 (h_1 h_2^{-1} p)$$

$\in g_1 H.$

So $g_2 H \subseteq g_1 H$

So $g_1 H = g_2 H.$

(c) To show:

$$\mathrm{Card}(G) = \mathrm{Card}(G/H)\,\mathrm{Card}(H).$$

Since

$$G = \bigcup_{gH \in G/H} gH \quad \text{then}$$

$$\mathrm{Card}(G) = \sum_{gH \in G/H} \mathrm{Card}(gH)$$

$$= \sum_{gH \in G/H} \mathrm{Card}(H)$$

$$= \mathrm{Card}(H)\Big(\sum_{gH \in G/H} 1\Big)$$

$$= \mathrm{Card}(H)\,\mathrm{Card}(G/H)$$

Corollary Let G be a group
Let H be a subgroup of G.
Then
Card(H) divides Card(G).

$$A_4 = \{ g \in S_4 \mid \cancel{\det}(g) = 1 \}$$

Grace's Theorem

If $g \in G$ and $h_1 \in H$
then $g h_1 H = gH$.

Proof Assume $g \in G$ and $h_1 \in H$.

To show: (a) $g h_1 H \subseteq gH$
(b) $gH \subseteq g h_1 H$.

Since $h_1 \in H$ then $g h_1 \in gH$.
Also $g h_1 = g h_1 \cdot 1 \in g h_1 H$.
So $gH \cap g h_1 H \neq \emptyset$. So $gH = g h_1 H$.