GTLA Lecture 21.08.2020

Theorem If $\mathbb{F}$ is algebraically closed and $A \in M_n(\mathbb{F})$ then A has an eigenvector over $\mathbb{F}$.

Example $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Does not have an eigenvector over $\mathbb{R}$.

It does have an eigenvector over $\mathbb{C}$ ($\mathbb{C}$ is alg. closed).

Proof of the theorem

Assume $\mathbb{F}$ is algebraically closed and $A \in M_n(\mathbb{F})$.

Since $\mathbb{F}$ is algebraically closed then $\det(x - A)$ has a root.

So there exists $\lambda \in \mathbb{F}$ such that

$$\det(\lambda - A) = 0.$$

So $\lambda - A$ is not invertible.

So $\ker(\lambda - A) \neq 0$.

So there exists $p \in \ker(\lambda - A)$

with $p \neq 0$.

So $(\lambda - A)p = 0$. (this is what kernel means)

So $Ap = \lambda p$.

So p is an eigenvector. //

Examples we did

① $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad m_{A_1}(x) = x-1$.

$\det(x - A_1) = (x-1)^2$

2 lin. indep. eigenvectors over $\mathbb{C}$.

$p_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $p_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Both with eigenvalue 1.

② $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad m_{A_2}(x) = (x-1)^2$

$\det(x - A_2) = (x-1)^2$

Only 1 lin. indep. eigenvector over $\mathbb{C}$.

$p_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ eigenvalue 1.

③ $A_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad m_{A_3}(x) = x^2 + 1$

$= (x-i)(x+i)$

$\det(x - A_3) = x^2 + 1$.

2 lin. indep. eigenvectors over $\mathbb{C}$

$p_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ $\quad p_2 \begin{pmatrix} 1 \\ -i \end{pmatrix}$

eigenvalue i eigenvalue $-i$.

No eigenvectors over $\mathbb{R}$.

Direct sums of matrices

i.e. Block diagonal matrices

$$A = A_1 \oplus A_2 \oplus A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \in M_6(\mathbb{C})$$

$$m_A(x) = (x-1)^2 (x-i)(x+i)$$

$$\det(x-A) = (x-1)^4 (x-i)(x+i)$$

5 linearly indep. eigenvectors

$$p_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad p_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad p_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

eigenvalue 1.   eigenvalue 1 eigenvalue 1

$$p_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ i \end{pmatrix} \quad p_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -i \end{pmatrix}$$

eigenvalue i eigenvalue $-i$.

$$A p_5 = \begin{pmatrix} 1 & 0 & & & & \\ 0 & 1 & & & & \\ & & 1 & 1 & & \\ & & 0 & 1 & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -i \end{pmatrix}$$

$$= i \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -i \end{pmatrix} = i p_5.$$

<u>Prop</u> Let $A_1 \in M_{n_1}(\mathbb{C})$ and $A_2 \in M_{n_2}(\mathbb{C})$. Then

$$A_1 \oplus A_2 \in M_{n_1+n_2}(\mathbb{C})$$

and

$$m_{A_1 \oplus A_2}(x) = \operatorname{lcm}(m_{A_1}(x), m_{A_2}(x))$$

$$\det(x - (A_1 \oplus A_2)) = \det(x - A_1)\det(x - A_2).$$

<u>Direct sums of subspaces</u>

Let $\mathbb{F}$ be a field and

V an $\mathbb{F}$ vector space.

$V = U \oplus W$ means

(a) U is a subspace of V

(b) W is a subspace of V

(c) $U + W = V$ $\quad U+W = \{u+w \mid u \in U \text{ and } w \in W\}$

(d) $U \cap W = \{0\}$. $\quad U \cap W = \{v \in V \mid v \in U \text{ and } v \in W\}$

<u>Proposition</u> Let $\mathbb{F}$ be a field and V an $\mathbb{F}$-vector space

Assume $V = U \oplus W$.

Let B be a basis U.

and C a basis W.

(a) $B \cup C$ is a basis of V.

(b) Let $f_1 : U \to U$ a lin. transformation and $f_2 : W \to W$ a lin. transf.

Define
$$f : U \oplus W \longrightarrow U \oplus W$$
$$u + w \longmapsto f_1(u) + f_2(w).$$

Let A_1 be the matrix of f_1 with respect to the basis B.

Let A_2 be the matrix of f_2 with respect to the basis C.

Then

(ba) f is a linear transformation

(bb) The matrix of f with resp. to $B \cup C$ is

$$\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right) = A_1 \oplus A_2$$

(bc) $\ker(f) = \ker(f_1) \oplus \ker(f_2)$

(look at Tute sheet 3, Qns. 5).

Let $A \in M_n(\mathbb{F})$. Then A is <u>diagonalisable</u> if there exists $P \in GL_n(\mathbb{F})$ and $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ such that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Let $A \in M_n(\mathbb{F})$ and $P \in GL_n(\mathbb{F})$.

$\boxed{P^{-1}AP \text{ is the "conjugate of } A \text{ by } P\text{"}}$

Proposition Let $A \in M_n(\mathbb{F})$ and $P \in GL_n(\mathbb{F})$. Then

(a) $\det(P^{-1}AP) = \det(A)$

$\Big(\det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P)$
$= \det(P)^{-1}\det(P)\det(A) = \det(A)\Big)$.

(b) $\det(x - P^{-1}AP) = \det(x - A)$

$\Big(\det(x - P^{-1}AP) = \det(P^{-1}xP - P^{-1}AP)$
$= \det(P^{-1}(x - A)P) = \det(x - A)\Big)$.

(c) $m_{P^{-1}AP}(x) = m_A(x)$.

If $q(x)$ is a poly.

$q(x) = a_0 + a_1 x + a_2 x^2$.

$q(A) = a_0 + a_1 A + a_2 A^2$

$q(P^{-1}AP) = a_0 + a_1 P^{-1}AP + a_2 P^{-1}A^2P$

$= P^{-1}(a_0 + a_1 A + a_2 A^2)P$

$= P^{-1}q(A)P$.

If $q(A) = 0$ then $q(P^{-1}AP) = 0$.

$$x = \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix}$$

$xP = Px$

$P^{-1}P = I$.

$$\boxed{\begin{array}{l} \det(x - A) \text{ is a multiple of } m_A(x). \\ \text{We want } m_A(A) = 0. \end{array}}$$