GTLA Lecture 25.08.2020

Jordan Normal Form

$\mathbb{C}$ algebraically closed field.

Jordan blocks Let $d \in \mathbb{Z}_{>0}$ and $\lambda \in \mathbb{C}$. The Jordan block of size d and eigenvalue λ is

$$J_d(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \in M_d(\mathbb{C})$$

Example $\begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$ is the Jordan block of size 4 with eigenvalue 5

Theorem (Jordan Normal Form)
Let $n \in \mathbb{Z}_{>0}$ and $A \in M_n(\mathbb{C})$. Then there exists $P \in GL_n(\mathbb{C})$ such that

$P^{-1}AP$ is a direct sum of Jordan blocks.

(the Jordan blocks for A are unique up to reordering).

Recall Direct sum of matrices.

$$A_1 \oplus A_2 = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right)$$

$$A_1 \oplus A_2 \oplus A_3 = \begin{pmatrix} A_1 & & 0 \\ & \boxed{A_2} & \\ 0 & & \boxed{A_3} \end{pmatrix}$$

One Jordan block

Example $J = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$ size 2 eigenvalue 5.

$$\det(x - J) = \det\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} - \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} x-5 & -1 \\ 0 & x-5 \end{pmatrix} = (x-5)^2.$$

If $J = \underbrace{\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}}_{d \text{ columns}} = J_d(\lambda)$ then

$\det(x-J) = (x-\lambda)^d.$

<u>Example</u> $J = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}.$

$m_J(x)$ divides $\det(x-J) = (x-5)^2$

So $m_J(x)$ is $(x-5)^2$ or $(x-5)$

Since $J-5 = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\neq 0$, so $m_J(x) \neq x-5.$

So $m_J(x) = (x-5)^2.$

If $J = J_d(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$

then

$m_J(x) = (x-\lambda)^d.$

<u>Example</u> $J = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}.$

Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

$Je_1 = 5e_1$ (eigenvector eigenvalue 5)

$Je_2 = 5e_2 + e_1.$

$(J-5) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and

$(J-5)e_1 = 0$

$(J-5)e_2 = e_1$

$e_2 \xrightarrow{J-5} e_1 \xrightarrow{J-5} 0.$

If $J = J_d(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \in M_d(\mathbb{C})$.

and $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ith spot.

($\{e_1, \dots, e_d\}$ is my favourite basis of $\mathbb{C}^d$).

$(J-\lambda)e_1 = 0$

$(J-\lambda)e_2 = e_1$

$(J-\lambda)e_3 = e_2$

$\vdots$

$(J-\lambda)e_d = e_{d-1}$

$e_d \xrightarrow{J-\lambda} e_{d-1} \xrightarrow{J-\lambda} e_{d-2} \to \cdots \to e_1 \to 0$

$(e_1, \dots, e_d)$ is the waterfall basis for J.

If $A \in M_d(\mathbb{C})$ and $P \in GL_d(\mathbb{C})$

$P^{-1}AP = J_d(\lambda)$ then there exists a basis $\{b_1, \dots, b_d\}$ of $\mathbb{C}^d$

such that

$$(A-\lambda)b_1 = 0$$
$$(A-\lambda)b_2 = b_1$$
$$\vdots$$
$$(A-\lambda)b_d = b_{d-1}$$

$$b_d \xrightarrow{A-\lambda} b_{d-1} \to \cdots \to b_1 \to 0$$

$\{b_1, \ldots, b_d\}$ is the "waterfall" basis for A.

$$\boxed{P = \begin{pmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_d \\ | & | & & | \end{pmatrix} \text{ so that } P^{-1}AP = J_d(\lambda).}$$

(diagonalisable matrices have n linearly independent eigen-vectors: "eigenvector bases").

One Jordan block case has a "waterfall basis" with one eigenvector.

An <u>eigenvector</u> is $p \in \ker(\lambda - A)$.

The waterfall basis come from $\ker(\lambda - A)$, $\ker((\lambda-A)^2)$, $\ker((\lambda-A)^3)$, ...

Direct sum of Jordan blocks with same eigenvalue

Let $\lambda \in \mathbb{C}$, let $d_1, \dots, d_k \in \mathbb{Z}_{>0}$.

$$J = J_{d_1}(\lambda) \oplus \cdots \oplus J_{d_k}(\lambda)$$

$$= \begin{pmatrix} \left.\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}\right\} d_1 & & & 0 \\ & \left.\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}\right\} d_2 & & \\ & & \ddots & \\ 0 & & & d_k \left\{\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}\right. \end{pmatrix}$$

__Recall__ If $A = A_1 \oplus A_2 = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right)$

then

$$\det(x - A) = \det(x - A_1)\det(x - A_2).$$

$$m_A(x) = \mathrm{lcm}(m_{A_1}(x), m_{A_2}(x)).$$

So $\det(x - J) = (x-\lambda)^{d_1}(x-\lambda)^{d_2}\cdots(x-\lambda)^{d_k}$

$$= (x-\lambda)^{d_1+d_2+\cdots+d_k}.$$

$$m_J(x) = \mathrm{lcm}\left((x-\lambda)^{d_1}, (x-\lambda)^{d_2}, \dots, (x-\lambda)^{d_k}\right)$$

$$= (x-\lambda)^{\max(d_1,\dots,d_k)}.$$

Characteristic Polynomial and minimal polynomial for a direct sum of Jordan blocks with same eigenvalue λ.

<u>One</u> eigenvector for each Jordan block.

Each Jordan block has it's own waterfall basis with <u>one</u> eigenvector.

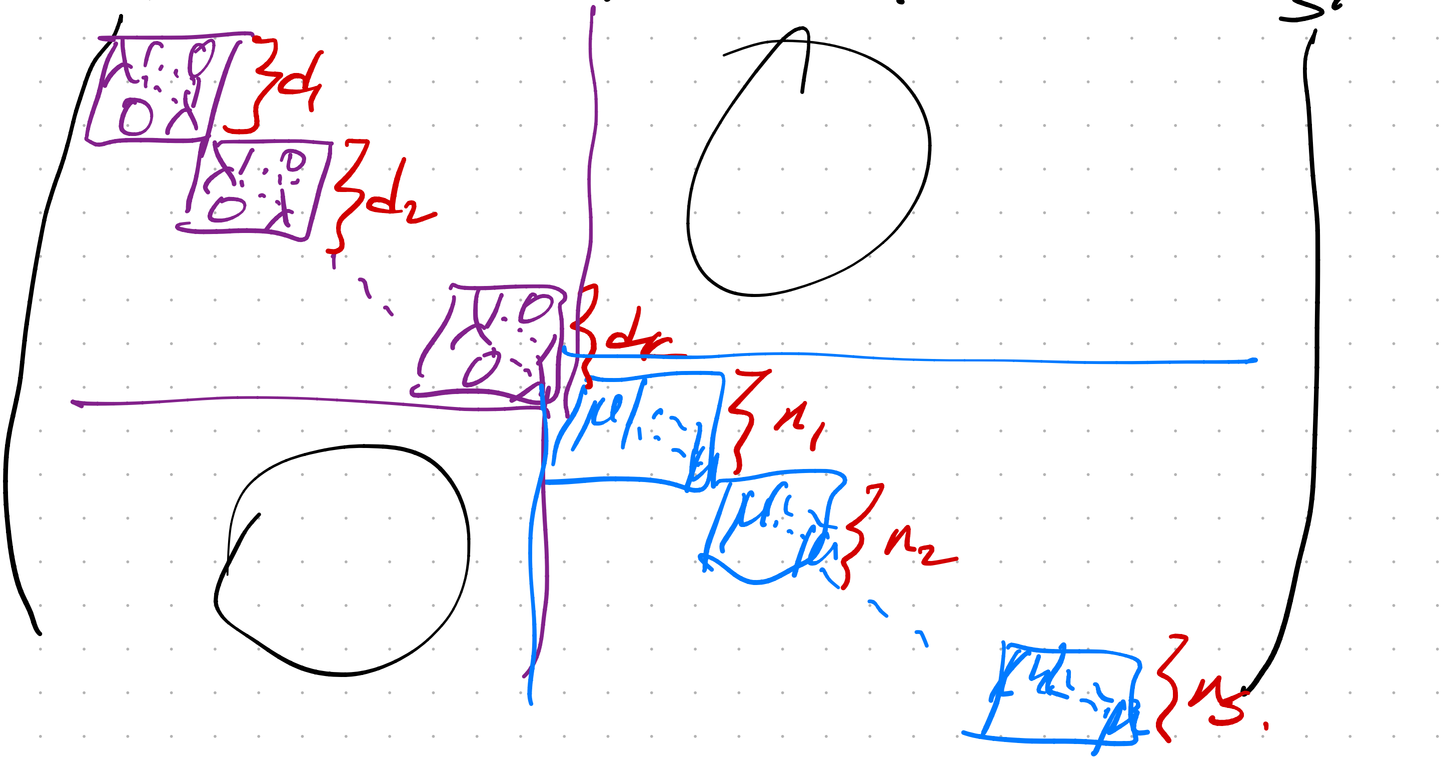
<u>Jordan blocks with different eigenvalues</u>

Let $\lambda \in \mathbb{C}$. Let $d_1, \dots, d_k \in \mathbb{Z}_{>0}$

Let $\mu \in \mathbb{C}$. Let $n_1, \dots, n_s \in \mathbb{Z}_{>0}$.

$$J = J_{d_1}(\lambda) \oplus \cdots \oplus J_{d_k}(\lambda) \oplus J_{n_1}(\mu) \oplus \cdots \oplus J_{n_s}(\mu)$$

$=$



$$\det(x-J) = \det\left(x-\left(J_{d_1}(\lambda)\oplus\cdots\oplus J_{d_k}(\lambda)\right)\right)\cdot\det\left(x-\left(J_{n_1}(\mu)\oplus\cdots\oplus J_{n_s}(\mu)\right)\right)$$

$$= (x-\lambda)^{d_1+\cdots+d_k}(x-\mu)^{n_1+\cdots+n_s}.$$

Assuming $\lambda\neq\mu$

$$m_J(x) = \operatorname{lcm}\left((x-\lambda)^{\max(d_1,\dots,d_k)},\ (x-\mu)^{\max(n_1,\dots,n_s)}\right)$$

$$= (x-\lambda)^{\max(d_1,\dots,d_k)}(x-\mu)^{\max(n_1,\dots,n_s)}.$$

Let

$$J = J_5(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ & \lambda & 1 & 0 & 0 \\ & & \lambda & 1 & 0 \\ & 0 & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$$

$$\lambda - J = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ & 0 & -1 & 0 & 0 \\ & & 0 & -1 & 0 \\ 0 & & & 0 & -1 \\ & & & & 0 \end{pmatrix}$$

Find $\ker(\lambda - J)$.
$= \operatorname{span}\{e_1\}$

$$(\lambda - J)^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 1 \\ & & & 0 & 0 \\ 0 & & & & 0 \end{pmatrix}$$

Find $\ker((\lambda-J)^2)$

$$(\lambda - J)^3 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ & 0 & 0 & 0 & -1 \\ & & 0 & 0 & 0 \\ & 0 & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

Find $\ker((\lambda-J)^3)$

$\ker(\lambda-J)^4 = \text{span}\{e_1, e_2, e_3, e_4\}$

$$(\lambda - J)^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & 0 & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

Find $\ker((\lambda-J)^4)$.

$(\lambda-J)^4 e_5 = e_1$

$(\lambda-J)^4 e_1 = 0$

$(\lambda-J)^4 e_2 = 0$

$(\lambda-J)^4 e_3 = 0$

$(\lambda-J)^4 e_4 = 0$

$\underline{(\lambda-J)}^5 = 0.$

A matrix A is <u>nilpotent</u> if there exists $l \in \mathbb{Z}_{>0}$ such that

$$A^l = 0.$$

$$\left(\begin{array}{c|c} \begin{matrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{matrix} \end{array}\right)$$

is also nilpotent

$$\boxed{A = \begin{pmatrix} 5 & 1 & 0 \\ & 5 & 1 \\ 0 & & 5 \end{pmatrix} \text{ } \underline{\text{not}} \text{ nilpotent}}$$

$5 - A$ is nilpotent. $\quad \underline{5-A} = \begin{pmatrix} 0 & -1 & 0 \\ & 0 & -1 \\ 0 & & 0 \end{pmatrix}$

$$\det \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array} \right)$$

$$= \det(A_1)\det(A_2)$$

$m_{A_1 \oplus A_2}$ is smallest poly.
s.t. $m_{A_1 \oplus A_2}(A_1 \oplus A_2) = 0$.

m_{A_1} is smallest s.t. $m_{A_1}(A_1) = 0$
m_{A_2} is smallest st $m_{A_2}(A_2) = 0$.

$$m_{A \oplus A_2}(A_1 \oplus A_2) = \left(\begin{array}{c|c} m_{A \oplus A_2}(A_1) & 0 \\ \hline 0 & m_{A \oplus A_2}(A_2) \end{array} \right)$$

Upper left
is 0 only if
$m_{A_1 \oplus A_2}$ is a multiple of m_{A_1}

Bottom right
need a multiple of m_{A_2}.