1.6 The binomial theorem

Let $k \in \mathbb{Z}_{\geq 0}$. Define k **factorial** by

$$0! = 1 \qquad \text{and} \qquad k! = k \cdot (k-1) \cdots 3 \cdot 2 \cdot 1 \text{ if } k \in \mathbb{Z}_{>0}.$$

Let $n, k \in \mathbb{Z}_{\geq 0}$ with $k \leq n$. Define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Theorem 1.1. *Let $n, k \in \mathbb{Z}_{\geq 0}$ with $k \leq n$.*

(a) Let S be a set with cardinality n. Then
$\binom{n}{k}$ is the number of subsets of S with cardinality k.

(b) $\binom{n}{k}$ is the coefficient of $x^{n-k}y^k$ in $(x+y)^n$.

(c) If $k \in \{1, \ldots, n-1\}$ then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \qquad \textit{and} \qquad \binom{n}{0} = 1 \quad \textit{and} \quad \binom{n}{n} = 1.$$

This theorem says that the table of numbers

$$\begin{array}{ccccccccccc}
&&&&& \binom{0}{0} &&&&& \\
&&&& \binom{1}{0} && \binom{1}{1} &&&& \\
&&& \binom{2}{0} && \binom{2}{1} && \binom{2}{2} &&& \\
&& \binom{3}{0} && \binom{3}{1} && \binom{3}{2} && \binom{3}{3} && \\
& \binom{4}{0} && \binom{4}{1} && \binom{4}{2} && \binom{4}{3} && \binom{4}{4} & \\
\binom{5}{0} && \binom{5}{1} && \binom{5}{2} && \binom{5}{3} && \binom{5}{4} && \binom{5}{5} \\
\ddots &&&&& \vdots &&&&& \ddots
\end{array}$$

are the numbers in **Pascal's triangle**

$$\begin{array}{ccccccccccc}
&&&&& 1 &&&&& \\
&&&& 1 && 1 &&&& \\
&&& 1 && 2 && 1 &&& \\
&& 1 && 3 && 3 && 1 && \\
& 1 && 4 && 6 && 4 && 1 & \\
1 && 5 && 10 && 10 && 5 && 1 \\
\ddots &&&&& \vdots &&&&& \ddots
\end{array}$$

and that

$$\begin{array}{rcc}
(x+y)^0 &=& 1, \\
(x+y)^1 &=& x+y, \\
(x+y)^2 &=& x^2 + 2xy + y^2, \\
(x+y)^3 &=& x^3 + 3x^2y + 3xy^2 + y^3, \\
(x+y)^4 &=& x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4, \\
(x+y)^5 &=& x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5, \\
\vdots && \vdots
\end{array}$$