1^{st} order constant coefficient ODEs (homogeneous).

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Let $D = \frac{d}{dx}$ and $\lambda \in \mathbb{C}$.

Solve: $(D-\lambda)y = 0$.

Solution: Our equation is $\left(\frac{d}{dx} - \lambda\right)y = 0$.

So $\frac{dy}{dx} - \lambda y = 0$.

So $\frac{dy}{dx} = \lambda y$

So $\frac{1}{y}\frac{dy}{dx} = \lambda$

So $\int \frac{1}{y}\frac{dy}{dx}\,dx = \int \lambda\,dx$

So $\int \frac{1}{y}\,dy = \int \lambda\,dx$

So $\log(y) = \lambda x + c_1$

So $y = e^{\lambda x + c_1}$
$= e^{\lambda x}e^{c_1} = Ce^{\lambda x}$

where C and c_1 are constants.

So the solutions to $(D-\lambda)y = 0$ are

$y = Ce^{\lambda x}$, where C is a constant.

<u>2nd order constant coefficient ODEs</u> (homogeneous, $\lambda_1 \neq \lambda_2$). A. Ram. 11.09.2022

Let

$$D = \frac{d}{dx}, \qquad A, B \in \mathbb{C}, \qquad \lambda_1, \lambda_2 \in \mathbb{C} \text{ with } \lambda_1 \neq \lambda_2$$

Since $(D-\lambda_1)Ae^{\lambda_1 x} = 0$ and $(D-\lambda_2)Be^{\lambda_2 x} = 0$

then

$$(D-\lambda_1)(D-\lambda_2)(Ae^{\lambda_1 x} + Be^{\lambda_2 x})$$

$$= (D-\lambda_2)(D-\lambda_1)Ae^{\lambda_1 x} + (D-\lambda_1)(D-\lambda_2)Be^{\lambda_2 x}$$

$$= (D-\lambda_2)\cdot 0 + (D-\lambda_1)\cdot 0 = 0 + 0 = 0.$$

So

$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$, where A and B are constants

are solutions to $(D-\lambda_1)(D-\lambda_2)y = 0$.

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Example 6.2 Solve $y''+7y'+12y=0$.

Solution. Let $D=\frac{d}{dx}$. Then $y''+7y'+12y=(D^2+7D+12)y$ and our equation is

$$(D^2+7D+12)y=0. \quad \text{So} \quad (D+3)(D+4)y=0.$$

So

$$y=Ae^{-3x}+Be^{-4x},$$ where A and B are constants

are solutions to $(D-(-3))(D-(-4))y=0$ (i.e. solutions to $y''+7y'+12y=0$).

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2nd order constant coefficient ODEs (homogeneous, $\lambda_1 = \lambda_2$)

Let
$$D = \frac{d}{dx}, \qquad A, B \in \mathbb{C}, \qquad \lambda \in \mathbb{C}.$$

Then
$$(D-\lambda)^2(Ae^{\lambda x} + Bxe^{\lambda x}) = (D-\lambda)\left((D-\lambda)Ae^{\lambda x} + (D-\lambda)Bxe^{\lambda x}\right)$$
$$= (D-\lambda)(0 + Bx\lambda e^{\lambda x} + Be^{\lambda x} - \lambda Bxe^{\lambda x})$$
$$= (D-\lambda)Be^{\lambda x} = 0.$$

So

$y = Ae^{\lambda x} + Bxe^{\lambda x}$, where A and B are constants

are solutions to $(D-\lambda)^2 y = 0$. ~~i.e. solutions to~~

<u>Example 6.3</u> Solve $y'' + 2y' + y = 0$.

<u>Solution</u>: Let $D = \frac{d}{dx}$. Then $y'' + 2y' + y = (D^2 + 2D + 1)y$

and our equation is

$$(D^2 + 2D + 1)y = 0. \qquad \text{So} \quad (D+1)^2 y = 0.$$

So $y = Ae^{-x} + Bxe^{-x}$, where A and B are constants

are solutions to $(D - (-1))^2 y = 0$ (i.e. solutions to

$y'' + 2y' + y = 0$).

2^{nd} order constant coefficient ODEs (homogeneous, $\lambda_2 = \bar{\lambda}_1$) A. Ram

Real valued solutions are determined as follows.

Let $D = \frac{d}{dx}$, $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1 = \bar{\lambda}_2$.

then $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ with $\alpha, \beta \in \mathbb{R}$.

If A and B are constants then

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x} = Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x}$$

$$= Ae^{\alpha x}e^{i\beta x} + Be^{\alpha x}e^{-i\beta x} = Ae^{\alpha x}(\cos\beta x + i\sin\beta x)$$
$$+ Be^{\alpha x}(\cos\beta x - i\sin\beta x)$$

$$= (A+B)e^{\alpha x}\cos\beta x + i(A-B)e^{\alpha x}\sin\beta x$$

$$= C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x,$$ where C_1, C_2 are constants.

So solutions of $(D-\lambda_1)(D-\lambda_2)y=0$ when $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$

are

$$y = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x,$$ where C_1 and C_2 are constants.

<u>Example 6.4</u> Solve $y'' - 4y' + 13y = 0$ if $y(0)=1$ and $y'(0)=6$.

<u>Solution.</u> Let $D = \frac{d}{dx}$. Then $y''-4y'+13y = (D^2-4D+13)y$

and our equation is

$$(D^2-4D+13)y=0.$$ Using the quadratic formula

$$\lambda_1 = \frac{4+\sqrt{4^2-4\cdot 13}}{2} = 2+\sqrt{4-13} = 2+\sqrt{-9} = 2+3i$$

$$\lambda_2 = \frac{4-\sqrt{4^2-4\cdot 13}}{2} = 2-\sqrt{4-13} = 2-\sqrt{-9} = 2-3i$$

and our equation is

$$(D-(2+3i))(D-(2-3i))y=0.$$

So

$y = C_1 e^{2x} \cos 3x + C_2 e^{2x} \sin 3x$, where C_1 and C_2 are constants

are solutions to

$(D^2 - 4D + 13)y = 0$ (i.e. solutions to $y'' - 4y' + 13y = 0$).

Plugging in $y(0) = 1$ gives

$$1 = C_1 e^{2\cdot 0} \cos 0 + C_2 e^{2\cdot 0} \sin 0 = C_1 e^{2\cdot 0} + C_2 e^{2\cdot 0}\cdot 0 = C_1 e^{2\cdot 0} + 0 = C_1 e^{2\cdot 0} = C_1$$

Plugging in $y'(0) = 6$ gives

$$6 = C_1(e^{2\cdot 0}(-\sin 3\cdot 0)\cdot 3 + 2e^{2\cdot 0}\cos(3\cdot 0)) + C_2(e^{2\cdot 0}(\cos 3\cdot 0)\cdot 3 + 2e^{2\cdot 0}\sin 3\cdot 0)$$

$$= C_1(1\cdot 0\cdot 3 + 2\cdot 1\cdot 1) + C_2(1\cdot 1\cdot 3 + 2\cdot 1\cdot 0)$$

$= 2C_1 + 3C_2$. Using $C_1 = 1$ gives $6 = 2 + 3C_2$ and $C_2 = \frac{4}{3}$.

So

$y = e^{2x}\cos 3x + \frac{4}{3} e^{2x} \sin 3x$ is a solution to

$y'' - 4y' + 13y = 0$ when $y(0) = 1$ and $y'(0) = 6$.