1.14 Limits

The *tolerance set* is

$$\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}.$$

For $n \in \mathbb{Z}_{>0}$ define $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$

$$d(x, y) = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}, \qquad \text{if } x = (x_1, \ldots, x_n) \text{ and } y = (y_1, \ldots, y_n).$$

Let $m, n \in \mathbb{Z}_{>0}$ and let $f\colon \mathbb{R}^m \to \mathbb{R}^n$. Let $a \in \mathbb{R}^m$ and $\ell \in \mathbb{R}^n$.

$$\lim_{x \to a} f(x) = \ell \qquad \text{means}$$

$$\text{if } \varepsilon \in \mathbb{E} \text{ then there exists } \delta \in \mathbb{E} \text{ such that} \quad \text{if } d(x, a) < \delta \text{ then } d(f(x), \ell) < \varepsilon.$$

Here is a translation into the language of "English":

In English	In Math
The client has a machine f that produces steel rods of length ℓ for sales.	Let $f\colon X \to \mathbb{R}$ and let $\ell \in \mathbb{R}$.
The output of f gets closer and closer to ℓ as the input gets closer and closer to a means	$\lim_{x \to a} f(x) = \ell$ means
if you give me a tolerance the client needs, in other words, the number of decimal places of accuracy the client requires	if $\varepsilon \in \mathbb{E}$
then my business will tell you	then there exists
the accuracy you need on the dials of the machine so that	$\delta \in \mathbb{E}$ such that
if the dials are set within δ of a	if $d(x, a) < \delta$
then the output of the machine will be within ε of ℓ.	then $d(f(x), \ell) < \epsilon$.

Let $n \in \mathbb{Z}_{>0}$ and let $a_1, a_2, \ldots$ be a sequence in $\mathbb{R}^n$. Let $\ell \in \mathbb{R}$.

$$\lim_{n \to \infty} a_n = \ell \qquad \text{means}$$

$$\text{if } \varepsilon \in \mathbb{E} \text{ then there exists } N \in \mathbb{Z}_{>0} \text{ such that} \quad \text{if } n \in \mathbb{Z}_{\geq N} \text{ then } d(a_n, \ell) < \varepsilon.$$