5.8 Limits and scalar multiplication: proof

Theorem 5.4. (Limits and scalar multiplication)
Let $n \in \mathbb{Z}_{>0}$. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ be functions and let $a \in \mathbb{R}^n$.

$$\textit{Assume that} \qquad \lim_{x\to a} f(x) \qquad \textit{exists.}$$

Then, if $c \in \mathbb{R}$ then $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$,

Proof.

Assume $c \in \mathbb{R}$ and let $l = \lim_{x\to a} f(x)$.
To show: $\lim_{x\to a} cf(x) = cl$.
To show: If $e \in \mathbb{Z}_{>0}$ then there exists $d \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^n \text{ is within } 10^{-d} \text{ of } a \text{ then } cf(x) \text{ is within } 10^{-e} \text{ of } cl.$$

Assume $e \in \mathbb{Z}_{>0}$.
Let $r \in \mathbb{Z}_{>0}$ be such that $c < 10^r$.
Since $l = \lim_{x\to a} f(x)$ then we know that there exists $d \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^n \text{ is within } 10^{-d} \text{ of } a \text{ then } f(x) \text{ is within } 10^{-(e+r)} \text{ of } l.$$

To show: If $x \in \mathbb{R}^n$ is within 10^{-d} of a then $cf(x)$ is within 10^{-e} of cl.
Assume $x \in \mathbb{R}^n$ is within 10^{-d} of a.
To show: $cf(x)$ is within 10^{-e} of cl.

$$d(cf(x), cl) = |cf(x) - cl| = |c| \cdot |f(x) - l| < |c| \cdot 10^{-(e+r)} < 10^r 10^{-(e+r)} = 10^{-e}.$$

So $cf(x)$ is within 10^{-e} of cl.
So $\lim_{x\to a} cf(x) = cl$.

□