1.8 Circular functions

Let $i \in \mathbb{C}$ be such that $i^2 = -1$. The *circular functions* $\sin z$ and $\cos z$ are given by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots \quad \text{and}$$
$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \cdots,$$

so that

$$e^{iz} = \cos z + i \sin z.$$

For $a, \theta \in \mathbb{R}$,

$$e^{a+i\theta} = e^a e^{i\theta} = re^{i\theta} = r(\cos\theta + i\sin\theta) = (r\cos\theta) + i(r\sin\theta) = x + iy,$$

are frequently (but not universally) used notational transformations. Define

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

1.9 Hyperbolic functions

The *hyperbolic functions* $\sinh z$ and $\cosh z$ are given by

$$\sinh z = \frac{e^{z} - e^{-z}}{2} = z + \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots \quad \text{and}$$
$$\cosh z = \frac{e^{z} + e^{-z}}{2} = 1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \cdots,$$

so that

$$e^{z} = \cosh z + \sinh z.$$

Define

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}, \quad \text{sech } x = \frac{1}{\cosh x}, \quad \text{csch } x = \frac{1}{\sinh x}.$$