02.03.2022

A. Ram ①

Some polynomials in $\mathbb{C}[x_1^{\pm1}, x_2^{\pm1}, x_0^{\pm1}]$

$E_{(0,0,0)} = 1, \qquad E_{(1,1,1)} = x_1x_2x_3 \qquad E_{(-1,-1,-1)} = x_1^{-1}x_2^{-1}x_3^{-1}$

$E_{(1,0,0)} = x_1$

$E_{(0,1,0)} = x_2 + \frac{1-t}{1-qt^2}x_1$

$E_{(0,0,1)} = x_3 + \frac{1-t}{1-qt}(x_2+x_1)$

$E_{(1,1,0)} = x_1x_2 \qquad E_{(-1,0,0)} = x_1^{-1} + \left(\frac{1-t}{1-qt}\right)(x_2^{-1}+x_3^{-1})$

$E_{(1,0,1)} = x_1x_3 + \frac{1-t}{1-qt^2}x_1x_2 \qquad E_{(0,-1,0)} = x_2^{-1} + \left(\frac{1-t}{1-qt^2}\right)x_3^{-1}$

$E_{(0,1,1)} = x_2x_3 + \frac{1-t}{1-qt}(x_1x_3+x_1x_2) \qquad E_{(0,0,-1)} = x_3^{-1}$

$P_{(0,0,0)} = 1 \qquad P_{(1,1,1)} = x_1x_2x_3$

$P_{(1,0,0)} = x_1+x_2+x_3 \qquad P_{(1,1,0)} = x_1x_2+x_1x_3+x_2x_3$

$P_{(2,0,0)} = x_1^2+x_2^2+x_3^2 + \frac{(1-q^2)(1-t)}{(1-tq)(1-q)}(x_1x_2+x_1x_3+x_2x_3)$

$P_{(3,0,0)} = m_3 + \frac{(1-q^3)(1-t)}{(1-q^2t)(1-q)}m_{21} + \frac{(1-q^3)(1-q^2)(1-t)^2}{(1-q^2t)(1-qt)(1-q)^2}m_{1^3}$

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$$E_{(2,1,0)} = x_1^2 x_2 + q\frac{(1-t)}{(1-qt^2)} x_1 x_2 x_3$$

$$P_{(2,1,0)} = m_{21} + \left(\frac{(1-t^2)(1-q^2t)}{(1-qt)(1-qt^2)} + \frac{(1-t)(1-q^2)}{(1-q)(1-qt)}\right) m_{1^3}$$

where

$$m_{21} = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + x_2^2x_3$$

$$m_{1^3} = x_1x_2x_3$$

$$A_{(2,1,0)} = x_2x_3^2 - tx_1x_3^2 - tx_2^2x_3 + t^2x_1x_2^2 + t^2x_1^2x_3 - t^3x_1^2x_2 + (t^2-t)x_1x_2x_3$$

Polynomials Let $n \in \mathbb{Z}_{>0}$.

$\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ has basis

$\{x^\mu \mid \mu \in \mathbb{Z}^n\}$ where $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$

for $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$

Mystery basis Let $q, t \in \mathbb{C}^\times$, e.g. $q = \frac{1}{3}, t = e^{-1/2}$

$\{E_\mu \mid \mu \in \mathbb{Z}^n\}$ is another basis of $\mathbb{C}[X]$.

Laziness:

$$E_\mu(x_1, \ldots, x_n; q, t) = E_\mu(q,t) = E_\mu.$$

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Symmetric group actions

$\mathbb{C}[X]$ is a vector space over $\mathbb{C}$

The symmetric group S_n acts on $\mathbb{C}[X]$ by permuting the variables.

If $i \in \{1, \ldots, n-1\}$ and $f \in \mathbb{C}[X]$ then

$$(s_i f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)$$

$\mathbb{Z}^n$ is a vector space over $\mathbb{Z}$

The symmetric group S_n acts on $\mathbb{Z}^n$ by permuting the coordinates.

If $i \in \{1, \ldots, n-1\}$ and $\mu \in \mathbb{Z}^n$ then

$$s_i(\mu_1, \ldots, \mu_n) = (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \mu_i, \mu_{i+2}, \ldots, \mu_n)$$

If $w \in S_n$ and $\mu = (\mu_1, \ldots, \mu_n)$ then

$$w x^\mu = x^{w\mu}$$

HW: Show that

$$\sum_{w \in S_n} w\left(\prod_{i<j} \frac{x_i - t x_j}{x_i - x_j}\right) = \frac{(1-t^n)(1-t^{n-1}) \cdots (1-t^2)(1-t)}{(1-t)^n}$$

$$= \prod_{i<j} \frac{1-t^{j-i+1}}{1-t^{j-i}}$$

03.02.2022
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Bosonic and Fermionic Macdonald polynomials

Let $\rho = (n-1, n-2, \ldots, 2, 1, 0)$

Define
$$(\mathbb{Z}^n)^+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\}$$
$$(\mathbb{Z}^n)^{++} = \{\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n \mid \gamma_1 > \cdots > \gamma_n\}.$$

HW: Show that
$$\begin{matrix} (\mathbb{Z}^n)^+ & \longrightarrow & (\mathbb{Z}^n)^{++} \\ \lambda & \longmapsto & \lambda+\rho \end{matrix}$$
is a bijection

Let $\lambda \in (\mathbb{Z}^n)^+$ so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

The bosonic Macdonald polynomial P_λ is
$$P_\lambda = \frac{1}{W_\lambda(t)} \sum_{w \in S_n} w\left(E_\lambda \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \right)$$
where $W_\lambda(t)$ is such that the coefficient of x^λ in P_λ is 1.

The fermionic Macdonald polynomial $A_{\lambda+\rho}$
$$A_{\lambda+\rho} = \left(\prod_{i<j} \frac{x_j - t x_i}{x_i - x_j} \right) \left(\sum_{w \in S_n} (-1)^{\ell(w)} w E_{\lambda+\rho} \right)$$
where
$$\ell(w) = \{(i,j) \mid i<j \text{ and } w(i) > w(j)\}$$

Classical symmetric functions

Proposition: If $\lambda_1 \geq \cdots \geq \lambda_n$ then $E_\lambda(0,t) = x^\lambda$.

The Hall-Littlewood polynomial is

$$P_\lambda(0,t) = \frac{1}{W_\lambda(t)} \sum_{w \in S_n} w\left(x^\lambda \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j} \right)$$

The Schur function is

$$s_\lambda = P_\lambda(0,0) = \sum_{w \in S_n} w\left(x^\lambda \prod_{i<j} \frac{x_i}{x_i - x_j} \right)$$

The monomial symmetric function is

$$m_\lambda = P_\lambda(0,1) = \frac{1}{W_\lambda(1)} \sum_{w \in S_n} w x^\lambda$$

The elementary symmetric function is

$$e_{\lambda'} = P_\lambda(1,0)$$

HW: Show that $e_{\lambda'} = P_\lambda(1,t)$

HW: Show that $m_\lambda = P_\lambda(q,1)$

HW: Show that $s_\lambda = P_\lambda(q,q)$.

The electronic Macdonald polynomials

Define $\pi: \mathbb{Z}^n \to \mathbb{Z}^n$ by

$$\pi(\mu_1, \ldots, \mu_n) = (\mu_n + 1, \mu_1, \ldots, \mu_{n-1})$$

Define $\partial_i : \mathbb{C}[X] \longrightarrow \mathbb{C}[X]$ by

$$\partial_i = \frac{1}{x_i - x_{i+1}}(1 - s_i), \quad \text{for } i \in \{1, \ldots, n-1\}$$

The <u>electronic Macdonald polynomials</u> E_μ are given by

(0) $E_{(0,\ldots,0)} = 1$

(1) $E_{\pi\mu} = q^{\mu_n} x_1 E_\mu(x_2, \ldots, x_n, q^{-1}x_1)$

(2) If $\mu_i > \mu_{i+1}$ then

$$E_{s_i\mu} = \left(\partial_i x_i - t x_i \partial_i + \frac{(1-t)q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}}{1 - q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}} \right) E_\mu$$

where $v_\mu \in S_n$ is minimal length such that $v_\mu \mu$ is weakly increasing.

<u>HW</u>: Show that if $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ then

$$E_{(\mu_1+1, \ldots, \mu_n+1)} = x_1 \cdots x_n E_\mu \quad \text{and}$$

$$E_{(\mu_1+r, \ldots, \mu_n+r)} = (x_1 \cdots x_n)^r E_\mu \quad \text{for } r \in \mathbb{Z}.$$

Creation formulas

Define operators $t_\pi^\vee$ and $t_1^\vee, \ldots, t_{n-1}^\vee$ by

$$(t_\pi^\vee E_\mu)(x_1, \ldots, x_n) = q^{\mu_n} x_1 E_\mu(x_2, \ldots, x_n, q^{-1}x_1)$$

$$t_i^\vee E_\mu = \left(\partial_i x_i - t x_i \partial_i + \frac{(1-t) q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}}{1 - q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}} \right) E_\mu$$

For $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ write

$(r, c) \in \mu$ if $r \in \{1, \ldots, n\}$ and $c \in \{1, \ldots, \mu_r\}$

The cylindrical coordinate of (r, c) is $r + nc$.

Define

$$u_\mu(r, c) = \#\{r' < r \mid (r', c) \in \mu\} + \#\{r' > r \mid (r', c-1) \in \mu\}$$

Then

$$E_\mu = \left(\prod_{(r,c) \in \mu} t^\vee_{u_\mu(r,c)} \cdots t_r^\vee t_1^\vee t_\pi^\vee \right) \cdot 1$$

with the product taken in increasing order by cylindrical coordinate.

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Bosonic and Fermionic symmetrizers

The **bosonic symmetrizer** is the operator $\mathbb{1}_0 : \mathbb{C}[X] \to \mathbb{C}[X]$ given by

$$\mathbb{1}_0 = t^{-\frac{1}{2}\ell(w_0)} \Big(\sum_{w \in S_n} w \Big) \Big(\prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \Big)$$

The **fermionic symmetrizer** is the operator $\varepsilon_0 : \mathbb{C}[X] \to \mathbb{C}[X]$ given by

$$\varepsilon_0 = t^{-\frac{1}{2}\ell(w_0)} \Big(\prod_{i<j} \frac{x_j - t x_j}{x_i - x_j} \Big) \Big(\sum_{w \in S_n} (-1)^{\ell(w)} w \Big)$$

The creation formulas for P_λ and $A_{\lambda+\rho}$

$$P_\lambda = \frac{t^{\frac{1}{2}\ell(w)}}{W_\lambda(t)} \mathbb{1}_0 E_\lambda \qquad A_{\lambda+\rho} = t^{\frac{1}{2}\ell(w_0)} \varepsilon_0 E_{\lambda+\rho}$$

HW: Show that $\ell(w_0) = \frac{1}{2} n(n-1)$