The finite Hecke algebra H_0

Let $n \in \mathbb{Z}_{>0}$ and $q, t^{\frac{1}{2}} \in \mathbb{C}^\times$.

A <u>permutation</u> w is a bijection $w: \{1, \dots, n\} \to \{1, \dots, n\}$

$$\mathrm{Inv}(w) = \{(i,j) \mid i<j \text{ and } w(i) > w(j)\}$$

$$\ell(w) = \#\mathrm{Inv}(w)$$

For $i \in \{1, \dots, n-1\}$ the <u>simple transposition</u> s_i is

$$\begin{matrix} s_i(i) = i+1 \\ s_i(i+1) = i \end{matrix} \quad \text{and} \quad s_i(j) = j \text{ for } j \notin \{i, i+1\}.$$

Then $\ell(s_i) = 1$ and

$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \text{ if } j \notin \{i-1, i+1\}.$$

A <u>reduced word for w</u> is an expression

$$w = s_{i_1} \cdots s_{i_\ell} \quad \text{with } \ell(w) = \ell(s_{i_1}) + \cdots + \ell(s_{i_\ell})$$

The <u>finite Hecke algebra</u> H_0 is generated by $C_{s_1}, \dots, C_{s_{n-1}}$ with relations

$T_1, \dots, T_{n-1}$

$$C_{s_i}^2 = (t^{\frac{1}{2}} + t^{-\frac{1}{2}}) C_{s_i}, \qquad C_{s_i} C_{s_j} = C_{s_j} C_{s_i} \text{ if } j \notin \{i-1, i+1\}$$

$$T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) T_i + 1 \qquad T_i T_j = T_j T_i$$

$$C_{s_i} C_{s_{i+1}} C_{s_i} + C_{s_{i+1}} = C_{s_{i+1}} C_{s_i} C_{s_{i+1}} + C_{s_i}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

You may choose blue or red. The conversion is $C_{s_i} = T_i + t^{-\frac{1}{2}}$

Symmetrizers

S_n is the group of permutations

H_0 is a t-deformation of $\mathbb{C}[S_n]$

$$\mathbb{C}[S_n] = \mathrm{span}\{w \mid w \in S_n\}$$

$$H_0 = \mathrm{span}\{T_w \mid w \in S_n\}$$

where $T_w = T_{i_1} \cdots T_{i_\ell}$ if $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word.

In $\mathbb{C}[S_n]$,

$$p_0 = \sum_{w \in S_n} w \quad \text{satisfies} \quad s_i p_0 = p_0$$

$$e_0 = \sum_{w \in S_n} (-1)^{\ell(w) - \ell(w_0)} w \quad \text{satisfies} \quad s_i e_0 = -e_0.$$

where $w_0 \in S_n$ is given by $w_0(i) = n - i$ so that

$Inv(w_0) = \{(i,j) \mid i < j\}$ and $\ell(w_0) = \frac{1}{2} n(n-1)$.

The bosonic and fermionic symmetrizers are

$$\mathbb{1}_0 = \sum_{w \in S_n} t^{\frac{1}{2}(\ell(w) - \ell(w_0))} T_w \quad \text{satisfies} \quad T_i \mathbb{1}_0 = t^{\frac{1}{2}} \mathbb{1}_0$$

$$\varepsilon_0 = \sum_{w \in S_n} (-t^{-\frac{1}{2}})^{\ell(w) - \ell(w_0)} T_w \quad \text{satisfies} \quad T_i \varepsilon_0 = -t^{-\frac{1}{2}} \varepsilon_0.$$

Push-pull operators

For $i \in \{1, \dots, n-1\}$ define operators on $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

$$\partial_i = \frac{1}{x_i - x_{i+1}}(1 - s_i) = (1+s_i)\frac{1}{x_i - x_{i+1}}$$ ordinary cohomology

$$D_{i+1,i} = (1+s_i)\frac{1}{1 - x_i^{-1}x_{i+1}}$$ K-theory

$$D_{i,i+1} = (1+s_i)\frac{1}{1 - x_i x_{i+1}^{-1}}$$ K-theory opposite Borel

$$C_{s_i} = (1+s_i)\frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}} x_i^{-1} x_{i+1}}{1 - x_i^{-1} x_{i+1}} = t^{-\frac{1}{2}} D_{i+1,i} - t^{\frac{1}{2}} D_{i,i+1}$$

The action of C_{s_i} gives an action of H_0 on $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Define

$$c_{ij}(x,t) = \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} x_i x_j^{-1}}{1 - x_i x_j^{-1}} = t^{-\frac{1}{2}}\left(\frac{x_j - t x_i}{x_j - x_i}\right)$$ and

$$c_w(x,t) = \prod_{(i,j) \in \mathrm{Inv}(w)} c_{ij}(x,t), \quad \text{for } w \in S_n$$

__Theorem__ As operators on $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$,

$$\mathbb{1}_0 = p_0 c_{w_0}(x^{-1}, t) = t^{-\frac{1}{2} l(w_0)} \left(\sum_{w \in S_n} w\right) \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j}.$$

$$\varepsilon_0 = c_{w_0}(x,t) e_0 = \left(\prod_{i<j} \frac{x_j - t x_i}{x_j - x_i}\right) t^{-\frac{1}{2} l(w_0)} \left(\sum_{w \in S_n} (-1)^{l(w) + l(w_0)} w\right)$$

<u>Example</u> $n=3$.

$S_3 = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$ with $s_i^2=1$ and $s_1s_2s_1 = s_2s_1s_2$.

$$p_0 = s_1s_2s_1 + s_1s_2 + s_2s_1 + s_1 + s_2 + 1$$

$$e_0 = s_1s_2s_1 - s_1s_2 - s_2s_1 + s_1 + s_2 - 1.$$

$H_0 = \mathrm{span}\{1, T_1, T_2, T_1T_2, T_2T_1, T_1T_2T_1\}$ with

$T_i^2 = (t^{\frac12} - t^{-\frac12})T_i + 1$ and $T_1T_2T_1 = T_2T_1T_2$.

Then

$$\mathbb{1}_0 = T_1T_2T_1 + t^{-1/2}(T_1T_2 + T_2T_1) + t^{-2/2}(T_1 + T_2) + t^{-3/2}$$

$$\varepsilon_0 = T_1T_2T_1 - t^{1/2}(T_1T_2 + T_2T_1) + t^{2/2}(T_1 + T_2) - t^{3/2}$$

<u>The electronic Macdonald polynomials</u> E_μ

$\{E_\mu \mid \mu \in \mathbb{Z}^n\}$ is a basis of $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm1}, \dots, x_n^{\pm1}]$

The E_μ are eigenvectors for the Cherednik-Dunkl operators $Y_1, \dots, Y_n$.

$$Y_i E_\mu = q^{-\mu_i} t^{(v_\mu(i)-1)+\frac12(n-1)} E_\mu.$$

where, for $\mu = (\mu_1, \dots, \mu_n)$

$$v_\mu(i) = \#\{r' < i \mid \mu_{r'} \le \mu_i\} + \#\{r' > i \mid \mu_{r'} < \mu_i\}$$

Defining E_μ recursively by $\tau_i^\vee$

For $i \in \{1, \dots, n-1\}$ let

$$\tau_i^\vee = c_{s_i} - \frac{t^{-\frac12} - t^{\frac12} y_i y_{i+1}^{-1}}{1 - y_i y_{i+1}^{-1}}$$

The E_μ are determined by $E_{(0,\dots,0)} = 1$,

(a) $E_{(\mu_n+1, \mu_1, \dots, \mu_{n-1})} = q^{-\mu_n} E_\mu (q^{-1} x_n, x_1, \dots, x_{n-1})$.

(b) If $\mu_i > \mu_{i+1}$ then $E_{s_i \mu} = t^{\frac12} \tau_i^\vee E_\mu$.

XY parallelism

$$c_{s_i} = \tau_i^\vee + \frac{t^{-\frac12} - t^{\frac12} y_i y_{i+1}^{-1}}{1 - y_i y_{i+1}^{-1}} = (\eta_{s_i} + 1) c_{i,i+1}(Y, t)$$

parallels

$$c_{s_i} = (1 + s_i) c_{i,i+1}(x^{-1}, t)$$

If $\eta_{s_i} E_\mu \neq 0$ then $\eta_{s_i}^2 E_\mu = E_\mu$ and

$\eta_{s_i} E_\mu = E_{s_i \mu}$ and $\eta_{s_i} E_{s_i \mu} = E_\mu$

and

$$\eta_{s_i} \eta_{s_{i+1}} \eta_{s_i} = \eta_{s_{i+1}} \eta_{s_i} \eta_{s_{i+1}}.$$

Theorem

$$\mathbb{1}_0 = p_0 c_{w_0}(x^{-1};t) = p_0^Y c_{w_0}(Y,t)$$

$$\varepsilon_0 = c_{w_0}(x,t)e_0 = c_{w_0}(Y^{-1},t)e_0^Y$$

where

$$p_0 = \sum_{w\in S_n} w \quad \text{and} \quad p_0^Y = \sum_{w\in S_n} \eta_w$$

$$e_0 = \sum_{w\in S_n}(-1)^{l(w)-l(w_0)} w \quad \text{and} \quad e_0^Y = \sum_{w\in S_n}(-1)^{l(w)-l(w_0)} \eta_w .$$

Let $\lambda\in\mathbb{Z}^n$ with $\lambda_1\geq\cdots\geq\lambda_n$. The bosonic and fermionic Macdonald polynomials are

$$P_\lambda = \frac{1}{W_\lambda(t)}\sum_{w\in S_n} w\Big(E_\lambda \prod_{i<j}\frac{x_i - tx_j}{x_i - x_j}\Big) = \frac{t^{\frac{1}{2}l(w_0)}}{W_\lambda(t)}\mathbb{1}_0 E_\lambda$$

and

$$A_{\lambda+\rho} = \Big(\prod_{i<j}\frac{x_j - tx_i}{x_j - x_i}\Big)\sum_{w\in W_0}(-1)^{l(w)-l(w_0)} wE_{\lambda+\rho} = t^{\frac{1}{2}l(w_0)}\varepsilon_0 E_{\lambda+\rho}$$

where $\rho = (n-1, n-2, \ldots, 2, 1, 0)$.

E-expansions

For $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ define homomorphisms $ev^t_\mu : \mathbb{C}[Y] \to \mathbb{C}[Y]$ by

$$ev^t_\mu(Y_i) = q^{-\mu_i} t^{-(v_\mu(i)-1)+\frac{1}{2}(n-1)} \qquad \text{for } i \in \{1, \ldots, n\}.$$

so that

if $f \in \mathbb{C}[Y]$ then $f E_\mu = ev^t_\mu(f) E_\mu$.

Theorem Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$.

Then

$$P_\lambda = \sum_{\mu \in S_n \lambda} \left(\prod_{\substack{i<j \\ \mu_i > \mu_j}} t \left(\frac{1 - q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j) - 1}}{1 - q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j)}} \right) \right) E_\mu$$

and

$$A_{\lambda+\rho} = \sum_{\mu \in S_n(\lambda+\rho)} \left(\prod_{\substack{i<j \\ \mu_i > \mu_j}} (-1) \left(\frac{1 - q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j) - 1}}{1 - q^{\mu_i - \mu_j} t^{v_\mu(i) - v_\mu(j)}} \right) \right) E_\mu$$