Principal Specializations

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Let $n \in \mathbb{Z}$ and $q, t^{\frac{1}{2}} \in \mathbb{C}^\times$, $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Let $\varepsilon_i = (0, \ldots, 0, \overset{i\text{th}}{1}, 0, \ldots 0) \in \mathbb{Z}^n$. Then

$$E_{\varepsilon_i} = E_{\varepsilon_i}(x_1, \ldots, x_n; q, t) = x_i + \frac{1-t}{1-qt^{n-(i-1)}}(x_{i-1} + \cdots + x_2 + x_1)$$

Define a homomorphism $ev_{pr} : \mathbb{C}[X] \to \mathbb{C}$ by

$$ev_{pr}(x_i) = t^{i-1}$$

Then

$$ev_{pr}(E_{\varepsilon_i}) = E_{\varepsilon_i}(1, t, t^2, \ldots, t^{n-1}; q, t)$$

$$= t^{i-1} + \frac{1-t}{1-qt^{n-(i-1)}}(t^{i-2} + \cdots + t + 1)$$

$$= t^{i-1} + \frac{(1-t)(1-t^{i-1})}{(1-qt^{n-(i-1)})(1-t)}$$

$$= \frac{t^{i-1} - qt^n + 1 - t^{i-1}}{1-qt^{n-(i-1)}} = \frac{1-qt^n}{1-qt^{n-(i-1)}}$$

Note: When $n=2$, $E_{(0,1)} = x_2 + \frac{1-t}{1-qt}x_1$ and

$$E_{(0,1)}(1,t) = t + \frac{1-t}{1-qt}\cdot 1 = \frac{1-qt^2}{1-qt}$$

$$E_{(0,1)}(t,1) = 1 + \frac{1-t}{1-qt}\cdot t = \frac{1-qt+t-t^2}{1-qt}.$$

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Three flavors of Macdonald polynomials Lecture 5 (2)

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(a) Electronic Macdonald polynomials E_μ

(b) Bosonic Macdonald polynomials P_λ

(c) Fermionic Macdonald polynomials $A_{\lambda+\rho}$

where
$$\mu=(\mu_1,\dots,\mu_n)\in\mathbb{Z}^n$$
$$\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n \text{ with } \lambda_1\geq\dots\geq\lambda_n,$$
$$\rho=(n-1,n-2,\dots,2,1,0).$$

Let $\mu=(\mu_1,\dots,\mu_n)\in\mathbb{Z}^n_{\geq 0}$. Write
$$(r,c)\in\mu \quad\text{if}\quad r\in\{1,\dots,n\} \text{ and } c\in\{1,\dots,\mu_r\}$$

Define
$$v_\mu(r)=\#\{r'<r \mid \mu_{r'}\leq\mu_r\}+\#\{r'>r\mid \mu_{r'}<\mu_r\}$$
$$u_\mu(r,c)=\#\{r'<r\mid (r',c)\notin\mu\}+\#\{r'>r\mid (r',c-1)\notin\mu\}.$$

Then $v_\mu\in S_n$ (min. length such that $v_\mu\mu$ is weakly increasing)

If $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n_{\geq 0}$ with $\lambda_1\geq\dots\geq\lambda_n$ define
$$n(\lambda)=\sum_{i=1}^{n}(i-1)\lambda_i \qquad n(\lambda)=\begin{array}{|c|c|c|c|}\hline 0&0&0&0\\\hline 1&1&1&1\\\hline 2&2&&\\\cline{1-2} 3&&&\\\cline{1-1} 4&&&\\\cline{1-1}\end{array}=15.$$

<u>Theorem</u> Let $\mu \in \mathbb{Z}^n_{\geq 0}$ and $\lambda \in \mathbb{Z}^n_{\geq 0}$ with $\lambda_1 \geq \cdots \geq \lambda_n$. Then

$$P_\lambda(1,t,t^2,\ldots,t^{n-1};q,t) = t^{n(\lambda)} \prod_{i<j} \prod_{k=1}^{\lambda_i-\lambda_j-1} \frac{1-q^k t^{j-i+1}}{1-q^k t^{j-i}}$$

and

$$E_\mu(1,t,t^2,\ldots,t^{n-1};q,t) = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \prod_{(r,c)\in\mu} \prod_{i=1}^{u_\mu(r,c)} \frac{1-q^{\mu_r-c+1} t^{v_\mu(r)-i+1}}{1-q^{\mu_r-c+1} t^{v_\mu(r)-i}}$$

and

$$A_{\lambda+\rho}(1,t,t^2,\ldots,t^{n-1};q,t) = 0.$$

The <u>creation formulas</u> are

$$E_\mu = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau^\vee_{u_\mu} \mathbb{1}_Y, \quad P_\lambda = \frac{t^{\frac{1}{2}\ell(w_0)}}{W_\lambda(t)} \mathbb{1}_0 E_\lambda, \quad A_{\lambda+\rho} = t^{\frac{1}{2}\ell(w_0)} \varepsilon_0 E_{\lambda+\rho}$$

intertwiners

symmetrizers

Identify polynomials with $\mathrm{Ind}_{H_Y}^{\tilde{H}}(\mathrm{triv}) = \tilde{H}\mathbb{1}_Y$

$\mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}] \xrightarrow{\sim} \tilde{H}\mathbb{1}_Y$

$x^\mu = x_1^{\mu_1}\cdots x_n^{\mu_n} \longmapsto X^\mu \mathbb{1}_Y$ with $T_i \mathbb{1}_Y = t^{\frac{1}{2}}\mathbb{1}_Y$

$E_\mu \longmapsto E_\mu \mathbb{1}_Y$ $\quad Y_i \mathbb{1}_Y = t^{-(i-1)+\frac{1}{2}(n-1)} \mathbb{1}_Y$

The $E_\mu \mathbb{1}_Y$ are eigenvectors

$$Y_i E_\mu = q^{-\mu_i} t^{-(v_\mu(i)-1)+\frac{1}{2}(n-1)} E_\mu.$$

The intertwiners satisfy

$$\tau_i^\vee Y_i = Y_{i+1}\tau_i^\vee$$

$$\tau_i^\vee Y_{i+1} = Y_i \tau_i^\vee \quad \text{and} \quad \tau_i^\vee Y_j = Y_j \tau_i^\vee \quad \text{if } j \notin \{i, i+1\}.$$

and $\tau_\pi^\vee Y_i = Y_{i+1}\tau_\pi^\vee$ where $Y_{i+n} = q^{-1} Y_i$.

so that

$$E_\mu = \Big(\prod_{(r,c)\in\mu} \tau^\vee_{u_\mu(r,c)} \cdots \tau_2^\vee \tau_1^\vee \tau_\pi^\vee \Big) \mathbb{1}_Y$$

In $\tilde{H}$,

$$\tau_i^\vee = T_i + t^{-\frac{1}{2}} \frac{(1-t)}{1 - Y_i^{-1} Y_{i+1}}, \qquad \tau_\pi^\vee = g^\vee$$

$$X_i = T_{i-1} \cdots T_1 g^\vee T_{n-1}^{-1} \cdots T_i^{-1}$$

Introduce $\mathbb{1}_X$ which satisfies

$$\mathbb{1}_X T_i = t^{\frac{1}{2}} \mathbb{1}_X, \qquad \mathbb{1}_X g^\vee = \mathbb{1}_X$$

$$\mathbb{1}_X X_i = t^{-\frac{1}{2}(n-1) + i - 1} \mathbb{1}_X.$$

So $\mathbb{1}_X E_\mu \mathbb{1}_Y = E_\mu\big(t^{-\frac{1}{2}(n-1)}, t^{-\frac{1}{2}(n-1)} t, \ldots, t^{-\frac{1}{2}(n-1)} t^{n-1}; q, t\big) \mathbb{1}_X \mathbb{1}_Y$

$$= t^{-\frac{1}{2}(n-1)|\mu|} E_\mu(1, t, t^2, \ldots, t^{n-1}; q, t) \mathbb{1}_X \mathbb{1}_Y$$

where $|\mu| = \mu_1 + \cdots + \mu_n$.

Use $\mathbb{1}_X \tau_\pi^\vee = \mathbb{1}_X q^\vee = \mathbb{1}_X$ and

$$\mathbb{1}_X \tau_i^\vee = \mathbb{1}_X\left(t^{\frac12} + \frac{t^{-\frac12}(1-t)}{1-Y_i^{-1}Y_{i+1}}\right) = \mathbb{1}_X \frac{t^{-\frac12} - t^{\frac12} Y_i^{-1}Y_{i+1}}{1-Y_i^{-1}Y_{i+1}} = \mathbb{1}_X\, c_{i+1,i}(Y).$$

Example

$$E_{\varepsilon_3}(x_1,x_2,x_3;q,t)\mathbb{1}_Y = t^{-\frac12 \ell(v_\mu^{-1})} \tau_2^\vee\tau_1^\vee\tau_\pi^\vee \mathbb{1}_Y = t^0 \tau_2^\vee \tau_1^\vee \tau_\pi^\vee \mathbb{1}_Y.$$

So

$$\begin{aligned}
\mathbb{1}_X E_{\varepsilon_3}\mathbb{1}_Y &= \mathbb{1}_X \tau_2^\vee\tau_1^\vee\tau_\pi^\vee\mathbb{1}_Y \\
&= \mathbb{1}_X c_{32}(Y)\tau_1^\vee\tau_\pi^\vee\mathbb{1}_Y \\
&= \mathbb{1}_X \tau_1^\vee c_{31}(Y)\tau_\pi^\vee \mathbb{1}_Y = \mathbb{1}_X\tau_1^\vee\tau_\pi^\vee c_{24}(Y)\mathbb{1}_Y \\
&= \mathbb{1}_X c_{21}(Y)\tau_\pi^\vee c_{24}(Y)\mathbb{1}_Y = \mathbb{1}_X \tau_\pi^\vee c_{14}(Y) c_{24}(Y)\mathbb{1}_Y \\
&= \mathbb{1}_X c_{14}(Y)c_{24}(Y)\mathbb{1}_Y = ev_0^t(c_{14}c_{24})\mathbb{1}_X\mathbb{1}_Y
\end{aligned}$$

where $ev_0^t: \mathbb{C}[Y]\to\mathbb{C}$ is the homomorphism given by

$$ev_0^t(Y_i) = t^{-(i-1)+\frac12(n-1)} \quad \text{and} \quad Y_{i+n} = q^{-1}Y_i.$$

Thus

$$E_\mu(1,t,t^2,\dots,t^{n-1};q,t) = t^{\frac12(n-1)|\mu|}\, t^{-\frac12\ell(v_\mu^{-1})} \prod_{(i,k)\in Inv(u_\mu)} ev_0^t(c_{ik}(Y)).$$

Since

$$\mathrm{Inv}(u_\mu) = \bigcup_{(r,c)\in\mu} \bigcup_{i=1}^{u_\mu(r,c)} \{ v_\mu(r),\ i + (\mu_r - c+1)n \}$$

and

$$ev_0^t\left(c_{(i,j+\ell n)}(Y)\right) = ev_0^t\left(\frac{t^{-\frac12} - t^{\frac12} Y_i Y_{j+\ell n}^{-1}}{1 - Y_i Y_{j+\ell n}^{-1}}\right)$$

$$= ev_0^t\left(\frac{t^{-\frac12} - t^{\frac12} q^\ell Y_i Y_j^{-1}}{1 - q^\ell Y_i Y_j^{-1}}\right) = t^{-\frac12}\left(\frac{1 - t q^\ell t^{j-i}}{1 - q^\ell t^{j-i}}\right)$$

then

$$E_\mu(1, t, t^2, \dots, t^{n-1}; q, t)$$

$$= t^{\frac12(n-1)|\mu|}\, t^{-\frac12 \ell(v_\mu^{-1})} \prod_{(r,c)\in\mu} \prod_{i=1}^{u_\mu(r,c)} t^{\frac12} \frac{1 - q^{\mu_r - c+1} t^{v_\mu(r)-i+1}}{1 - q^{\mu_r - c+1} t^{v_\mu(r)-i}}$$

$$= t^{m} \prod_{(r,c)\in\mu} \frac{(1-q^A t^{v_\mu(r)})}{(1-\cancel{q^A t^{v_\mu(r)-1}})} \frac{(1-\cancel{q^A t^{v_\mu(r)-1}})}{(1-\cancel{q^A t^{v_\mu(r)-2}})} \cdots \frac{(1-\cancel{q^A t^{v_\mu(r)-u_\mu(r,c)+1}})}{(1-q^A t^{v_\mu(r)-u_\mu(r,c)})}$$

$$= t^{m} \prod_{(r,c)\in\mu} \frac{1-q^{\mu_r-c+1} t^{v_\mu(r)}}{1-q^{\mu_r-c+1} t^{v_\mu(r)-u_\mu(r,c)}}$$