Orthogonality

$$\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}; q, t]$$

Define $\bar{\ }: \mathbb{C}[X] \to \mathbb{C}[X]$ by

$$\overline{f(x_1, \ldots, x_n; q, t)} = f(x_1^{-1}, \ldots, x_n^{-1}; q^{-1}, t^{-1}).$$

Let

$$(a; z)_k = (1-a)(1-za)(1-z^2a)\cdots(1-z^{k-1}a)$$

$$(a; z)_\infty = (1-a)(1-za)(1-z^2a)\cdots$$

Define

$$\nabla_{q,t} = \prod_{i \neq j} \frac{(x_i x_j^{-1}; q)_\infty}{(t x_i x_j^{-1}; q)_\infty} \quad \text{and} \quad \Delta_{q,t} = \nabla_{q,t} \cdot \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j}.$$

Define $(\,,\,)_{q,t}: \mathbb{C}[X] \times \mathbb{C}[X] \to \mathbb{C}$ by

$$(f_1, f_2)_{q,t} = ct(f_1 \bar{f_2} \Delta_{q,t})$$

where $ct(f)$ is the coefficient of x^0 in f.

The <u>adjoint</u> of a linear operator $M: \mathbb{C}[X] \to \mathbb{C}[X]$ is $M^*: \mathbb{C}[X] \to \mathbb{C}[X]$ given by

$$(Mf_1, f_2)_{q,t} = (f_1, M^* f_2)_{q,t}$$

for $f_1, f_2 \in \mathbb{C}[X]$.

Adjoints

Define operators $X_1,\dots,X_n$, $T_1,\dots,T_{n-1}$ and T_π by

X_i is multiplication by x_i,

$$T_j = -t^{-\frac{1}{2}} + (1+s_i)\frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} x_i^{-1} x_{i+1}}{1 - x_i^{-1} x_{i+1}}$$

$$(T_\pi f)(x_1,\dots,x_n) = f(q^{-1}x_n, x_1,\dots,x_{n-1}).$$

Proposition

$$X_i^* = X_i^{-1}, \quad T_j^* = T_j^{-1} \quad \text{and} \quad T_\pi^* = T_\pi^{-1}$$

The Cherednik-Dunkl operators are $Y_1,\dots,Y_n$ given by

$$Y_1 = T_\pi T_{n-1}\cdots T_2 T_1 \quad \text{and} \quad Y_{j+1} = T_j^{-1} Y_j T_j^{-1}.$$

The intertwiners are $\tau_1^\vee,\dots,\tau_n^\vee$ and $\tau_\pi^\vee$ given by

$$\tau_j^\vee = T_j + \frac{t^{-\frac{1}{2}}(1-t)}{1 - Y_j^{-1} Y_{j+1}} \quad \text{and} \quad \tau_\pi^\vee = X_1 T_1 \cdots T_{n-1}.$$

Proposition

$$(\tau_j^\vee)^* = \tau_j^\vee, \quad Y_i^* = Y_i^{-1} \quad \text{and} \quad (\tau_\pi^\vee)^* = (\tau_\pi^\vee)^{-1}.$$

Macdonald polynomials

Electronic $E_\mu = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau^\vee_{u_\mu} 1 = (\mathrm{const})\, \tau^\vee_{i_1} \cdots \tau^\vee_{i_\ell} 1$

Bosonic $P_\lambda = \dfrac{t^{\frac{1}{2}\ell(w_0)}}{W_\lambda(t)} \mathbb{1}_0 E_\lambda = (\mathrm{const})\, \mathbb{1}_0 E_\lambda$

Fermionic $A_{\lambda+\rho} = t^{\frac{1}{2}\ell(w_0)} \varepsilon_0 E_{\lambda+\rho} = (\mathrm{const})\, \varepsilon_0 E_{\lambda+\rho}$

where
$$\mathbb{1}_0 = \sum_{w \in S_n} t^{\frac{1}{2}(\ell(w)-\ell(w_0))} T_w \quad \text{and} \quad \varepsilon_0 = \sum_{w \in S_n} (-t^{\frac{1}{2}})^{\ell(w)-\ell(w_0)} T_w .$$

Key point:
$$Y_i E_\mu = q^{-\mu_i} t^{-(v_\mu(i)-1)+\frac{1}{2}(n-1)} E_\mu$$
$$= (\text{eigenvalue}) E_\mu .$$

E-expansions

$$P_\lambda = \sum_{\mu \in S_n \lambda} \left(\prod_{\substack{i<j \\ \mu_i > \mu_j}} t \, \frac{1 - q^{\mu_i-\mu_j} t^{v_\mu(i)-v_\mu(j)+1}}{1 - q^{\mu_i-\mu_j} t^{v_\mu(i)-v_\mu(j)}} \right) E_\mu$$

$$A_{\lambda+\rho} = \sum_{\mu \in S_n(\lambda+\rho)} \left(\prod_{i<j} (-1) \frac{1 - q^{\mu_i-\mu_j} t^{v_\mu(i)-v_\mu(j)-1}}{1 - q^{\mu_i-\mu_j} t^{v_\mu(i)-v_\mu(j)}} \right) E_\mu .$$

Orthogonality

Proposition

(a) If $\mu \neq \nu$ then $(E_\mu, E_\nu)_{q,t} = 0$,

(b) If $\lambda \neq \gamma$ then $(P_\lambda, P_\gamma)_{q,t} = 0$,

(c) If $\lambda \neq \gamma$ then $(A_{\lambda+\rho}, A_{\gamma+\rho})_{q,t} = 0$.

Idea of proof:

$$\begin{pmatrix}\text{eigen}\\ \text{value}\end{pmatrix}(E_\mu, E_\nu)_{q,t} = (Y_i E_\mu, E_\nu)_{q,t} = (E_\mu, Y_i^* E_\nu)_{q,t}$$

$$= (E_\mu, Y_i^{-1} E_\nu)_{q,t} = \begin{pmatrix}\text{different}\\ \text{eigen}\\ \text{value}\end{pmatrix}(E_\mu, E_\nu)_{q,t}. \quad //$$

Reduction of norms

Use

$$(\tau_i^\vee)^* = \tau_i^\vee \quad \text{and} \quad (\tau_i^\vee)^2 = \frac{(1-tY_iY_{i+1}^{-1})(1-tY_i^{-1}Y_{i+1})}{(1-Y_iY_{i+1}^{-1})(1-Y_i^{-1}Y_{i+1})}$$

$$= c_{i,i+1}(Y)\, c_{i,i+1}(Y^{-1})$$

$$\mathbb{1}_0^* = \mathbb{1}_0 \quad \text{and} \quad \mathbb{1}_0^2 = \Big(\sum_{w \in W_0} t^{\ell(w)}\Big)\mathbb{1}_0 = W_0(t)\mathbb{1}_0$$

$$\varepsilon_0^* = \varepsilon_0 \quad \text{and} \quad \varepsilon_0^2 = \Big(\sum_{w \in W_0} t^{\ell(w)}\Big)\varepsilon_0 = (-1)^{\ell(w_0)} W_0(t)\varepsilon_0.$$

to get

$$(E_\mu, E_\mu)_{q,t} = \left((\mathrm{const})\, \tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee \cdot 1, (\mathrm{const})\, \tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee \cdot 1\right)_{q,t}$$

$$= (\mathrm{const}) \left(\tau_{i_\ell}^\vee \cdots \tau_{i_1}^\vee \tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee 1, 1\right)_{q,t}$$

$$= (\mathrm{const}) \left(\tau_{u_\mu}(Y) \tau_{u_\mu}(Y^{-1}) \cdot 1, 1\right)_{q,t}$$

$$= (\mathrm{const}) \left(\tau_{u_\mu}\left(\begin{smallmatrix}\text{eigen}\\ \text{value}\end{smallmatrix}\right) \tau_{u_\mu}\left(\begin{smallmatrix}\text{eigen}\\ \text{value}^{-1}\end{smallmatrix}\right) \cdot 1, 1\right)_{q,t}$$

$$= \left(\prod_{(r,c)\in\mu} \prod_{i=1}^{u_\mu(r,c)} \frac{1 - q^{\mu_r - c+1} t^{\nu_\mu(r) - i+1}}{1 - q^{\mu_r - c+1} t^{\nu_\mu(r) - i-1}} \right) (1,1)_{q,t}.$$

$$(P_\lambda, P_\lambda)_{q,t} = \left((\mathrm{const})\, \mathbb{1}_0 E_\lambda, (\mathrm{const})\, \mathbb{1}_0 E_\lambda\right)_{q,t}$$

$$= (\mathrm{const}) (\mathbb{1}_0 E_\lambda, \mathbb{1}_0 E_\lambda)_{q,t} = (\mathrm{const}) (\mathbb{1}_0^* \mathbb{1}_0 E_\lambda, E_\lambda)_{q,t}$$

$$= (\mathrm{const}) (\mathbb{1}_0^2 E_\lambda, E_\lambda)_{q,t} = (\mathrm{const}) (\mathbb{1}_0 E_\lambda, E_\lambda)_{q,t}$$

$$= (\mathrm{const}) (P_\lambda, E_\lambda)_{q,t} = \frac{W_0(t)}{W_\lambda(t)} \left(\prod_{i<j} \frac{1 - q^{\lambda_i - \lambda_j} t^{j-i-1}}{1 - q^{\lambda_i - \lambda_j} t^{j-i}} \right) (E_\lambda, E_\lambda)_{q,t}$$

and

$$(A_{\lambda+\rho}, A_{\lambda+\rho})_{q,t} = W_0(t) \left(\prod_{i<j} \frac{1 - q^{\lambda_i - \lambda_j + j - i} t^{j-i+1}}{1 - q^{\lambda_i - \lambda_j + j - i} t^{j-i}} \right) (E_{\lambda+\rho}, E_{\lambda+\rho})_{q,t}.$$

Comparing levels

Using the "raising the level" and the "Weyl character formula" gives

Theorem

$$\frac{(P_\lambda(q,qt),P_\lambda(q,qt))_{q,qt}}{(P_{\lambda+\rho}(q,t),P_{\lambda+\rho}(q,t))_{q,t}} = \frac{W_0(qt)}{W_0(t^{+1})}\left(\prod_{i<j}\frac{1-q^{\lambda_i-\lambda_j+j-i}t^{j-i+1}}{1-q^{\lambda_i-\lambda_j+j-i}t^{j-i+1}}\right)$$

The recursion:

The base case: $t=1$. Then

$$\nabla_{q,1} = \prod_{i\neq j}\frac{(x_i x_j^{-1};q)}{(x_i x_j^{-1};q)} = 1 \quad \text{and}$$

$$\Delta_{q,1} = \nabla_{q,1}\cdot\prod_{i<j}\frac{1-1x_i^{-1}x_j^{+1}}{1-x_i^{-1}x_j^{+1}} = 1\cdot 1 = 1.$$

Then $P_\lambda(q,1)=m_\lambda$ and $(m_\lambda,m_\lambda)_{q,1}=W_\lambda(1)$.

So

$$\frac{(P_\lambda(q,q),P_\lambda(q,q))_{q,q}}{(m_{\lambda+\rho},m_{\lambda+\rho})_{q,1}} = \frac{W_0(q)}{W_0(1)}\prod_{i<j}\frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{\lambda_i-\lambda_j+j-i}} = W_0(q)\cdot 1.$$

and the recursion gives

<u>Theorem</u> Let $k \in \mathbb{Z}_{\geq 0}$ and let $t = q^k$. Then

$$(P_\lambda(q,q^k), P_\lambda(q,q^k))_{q,q^k} = W_0(q^k) \prod_{i<j} \prod_{r=1}^{k-1} \frac{1-q^{\lambda_i-\lambda_j+r} t^{j-i}}{1-q^{\lambda_i-\lambda_j-r} t^{j-i}}.$$

The general formula is

$$(P_\lambda(q,t), P_\lambda(q,t))_{q,t} = \prod_{i<j} \frac{(q^{\lambda_i-\lambda_j} t^{j-i};q)_\infty}{(t q^{\lambda_i-\lambda_j} t^{j-i};q)_\infty} \frac{(q^{\lambda_i-\lambda_j+1} t^{j-i};q)_\infty}{(q^{\lambda_i-\lambda_j+1} t^{j-i-1};q)_\infty}$$

The special case when $\lambda = 0$ gives Macdonald's constant term conjecture

$$(1,1)_{q,t} = \prod_{i<j} \frac{(t^{j-i};q)_\infty}{(t^{j-i+1};q)_\infty} \frac{(q t^{j-i};q)_\infty}{(q t^{j-i-1};q)_\infty} \quad \text{and}$$

$$(1,1)_{q,q^k} = \prod_{i=2}^{n} \begin{bmatrix} ik \\ k \end{bmatrix} \quad \text{where}$$

$$[k] = \frac{1-q^k}{1-q} \quad \text{and} \quad [k]! = [1][2]\cdots[k-1][k].$$