27.04.2021
Lecture 4 ①
A. Ram

Specializations: Vocabulary

Bosonic

$m_\lambda = P_\lambda(0,1)$ monomial

$m_\lambda = P_\lambda(q,1)$ monomial

$P_\lambda^{(\alpha)} = \lim_{t\to 1} P_\lambda(t^\alpha, t)$ Jack

$P_\lambda(0,t)$ Hall-Littlewood

$s_\lambda = P_\lambda(t,t)$ Schur

$e_{\lambda'} = P_\lambda(1,t)$ elementary

$s_\lambda = P_\lambda(0,0)$ Schur

$P_\lambda(q,0)$ q-Whittaker

$e_{\lambda'} = P_\lambda(1,0)$ elementary

Electronic

$E_\mu(1,\infty)$

$E_\mu(q,\infty)$ dual Iwahori Whittaker

$E_\mu(\infty,\infty)$ Demazure atoms

$E_\mu(1,t)$

$E_\mu(t,t)$

$E_\mu(\infty,t)$ dual Iwahori spherical

$E_\mu(q,1)$

$E_\mu(q,1)$

$E_\mu^{(\alpha)}$

$E_\mu(q,1)$

$E_\mu(\infty,1)$

$E_\mu(0,t)$ Iwahori spherical

$E_\mu(t,t)$

$E_\mu(1,t)$

$E_\mu(0,0)$ Key polynomials

$E_\mu(q,0)$ Iwahori Whittaker

$E_\mu(1,0)$

Examples to look at

$$E_{\varepsilon_i}(x_1,\dots,x_n;q,t) = x_i + \frac{1-t}{1-qt^{n-(i-1)}}(x_{i-1}+\cdots+x_1)$$

$$= x_i + q^{-1}t^{-(n-i)}\frac{(1-t^{-1})}{1-q^{-1}t^{-(n-i+1)}}(x_{i-1}+\cdots+x_1)$$

$$E_{(2,1,0)}(x_1,x_2,x_3;q,t) = x_1^2x_2 + q\frac{(1-t)}{1-qt^2}x_1x_2x_3$$

$$= x_1^2x_2 + t^{-1}\frac{1-t^{-1}}{1-q^{-1}t^{-2}}x_1x_2x_3$$

$$P_{(2,1,0)}(x_1,x_2,x_3;q,t) = m_{(2,1,0)} + \left(\frac{(1-t^2)(1-q^2t)}{(1-qt)(1-qt^2)} + \frac{(1-t)(1-q^2)}{(1-q)(1-qt)}\right)m_{(1^3)}$$

$$= e_2e_1 - \frac{(1-q)(1-t^3)}{(1-t)(1-qt^2)}e_3$$

where

$$m_{(2,1,0)} = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2$$

$$m_{(1^3)} = x_1x_2x_3 = e_3$$

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3, \quad e_1 = x_1+x_2+x_3.$$

Demazure operators

$$D_{i,i+1} = (1+s_i)\frac{1}{1-x_ix_{i+1}^{-1}} \quad \text{and} \quad D_{i+1,i} = (1+s_i)\frac{1}{1-x_i^{-1}x_{i+1}}$$

Specializations of intertwiners

The symmetric group S_n acts on $\mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$

$$(s_i f)(x_1,\ldots,x_n) = f(x_1,\ldots,x_{i-1},x_{i+1},x_i,x_{i+2},\ldots,x_n).$$

for $i\in\{1,\ldots,n-1\}$. Define operators $y_1,\ldots,y_n$ on $\mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$ by

$$(y_i f)(x_1,\ldots,x_n) = f(x_1,\ldots,x_{i-1},q^{-1}x_i,x_{i+1},\ldots,x_n).$$

The operators

$$T_\pi = s_1\cdots s_{n-1}y_n \quad\text{and}$$

$$T_i = t^{\frac12}D_{i,i+1} + t^{-\frac12}(D_{i+1,i}-1) \quad \text{for } i\in\{1,\ldots,n-1\}$$

are used to build the <u>Cherednik-Dunkl operators</u> $Y_1,\ldots,Y_n$ given by

$$Y_1 = T_\pi T_1\cdots T_{n-1} \quad\text{and}\quad Y_j = T_{j-1}^{-1}Y_{j-1}T_{j-1}^{-1}.$$

The <u>intertwiners</u> $\tau_\pi^\vee$ and $\tau_1^\vee,\ldots,\tau_{n-1}^\vee$ are

$$\tau_\pi^\vee = x_1T_1\cdots T_{n-1} = x_1T_\pi Y_1^{-1} = x_1s_1\cdots s_n y_n Y_1^{-1}$$

and
$$t^{\frac12}\tau_i^\vee = D_{i+1,i} + t(D_{i,i+1}-1) + \frac{(1-t)Y_i^{-1}Y_{i+1}}{1-Y_i^{-1}Y_{i+1}},$$

$$t^{-\frac12}\tau_i^\vee = D_{i,i+1} + t^{-1}(D_{i+1,i}-1) + \frac{(1-t^{-1})Y_iY_{i+1}^{-1}}{1-Y_iY_{i+1}^{-1}}.$$

These give

$$t^{\frac{1}{2}}\tau_i^{\vee}\Big|_{t=0} = D_{i+1,i} \quad \text{and} \quad t^{-\frac{1}{2}}\tau_i^{\vee}\Big|_{t=\infty} = D_{i,i+1}$$

when $\left.\dfrac{(1-t)y_i^{-1}y_{i+1}}{1-y_i^{-1}y_{i+1}}\right|_{t=0} = 0$ and $\left.\dfrac{(1-t^{-1})y_i y_{i+1}^{-1}}{1-y_i y_{i+1}^{-1}}\right|_{t=\infty} = 0$, respectively.

The formulas

$$t^{\frac{1}{2}}\tau_i^{\vee} = t^{\frac{1}{2}}T_i^{-1} + \frac{(1-t)y_i^{-1}y_{i+1}}{1-y_i^{-1}y_{i+1}} \quad \text{and}$$

$$t^{-\frac{1}{2}}\tau_i^{\vee} = t^{-\frac{1}{2}}T_i + \frac{(1-t^{-1})y_i y_{i+1}^{-1}}{1-y_i^{+1}y_{i+1}^{-1}}$$

give

$$t^{\frac{1}{2}}\tau_i^{\vee}\Big|_{q=0} = t^{\frac{1}{2}}T_i^{-1} \quad \text{and} \quad t^{-\frac{1}{2}}\tau_i^{\vee}\Big|_{q=\infty} = t^{-\frac{1}{2}}T_i$$

when $\left.\dfrac{(1-t)y_i^{-1}y_{i+1}}{1-y_i^{-1}y_{i+1}}\right|_{q=0} = 0$ and $\left.\dfrac{(1-t^{-1})y_i y_{i+1}^{-1}}{1-y_i y_{i+1}^{-1}}\right|_{q=\infty} = 0$, respectively.

Demazure characters for the affine Lie algebra

$$s_{\lambda,w} = D_w x^\lambda \quad \text{for } \lambda \in \mathfrak{h}^*_{\mathbb{Z}} \text{ and } w \in W.$$

The integral weight lattice for the affine Lie algebra $(\mathfrak{gl}_n \otimes \mathbb{C}[\varepsilon, \varepsilon^{-1}]) \oplus \mathbb{C}K \oplus \mathbb{C}d$ is

$$\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\delta, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \Lambda_0\},$$

where we may view $\delta, \varepsilon_1, \ldots, \varepsilon_n, \Lambda_0$ as formal symbols. Define operators $s_0, s_1, \ldots, s_{n-1}$ on $\mathfrak{h}^*_{\mathbb{Z}}$ by

$$s_i \lambda = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i \quad \text{for } i \in \{0, 1, \ldots, n-1\},$$

where

$$\alpha_0 = \varepsilon_n - \varepsilon_1 + \delta, \qquad \alpha_i = \varepsilon_i - \varepsilon_{i+1},$$

$$\alpha_0^\vee = \varepsilon_n^\vee - \varepsilon_1^\vee + K, \qquad \alpha_i^\vee = \varepsilon_i^\vee - \varepsilon_{i+1}^\vee,$$

$$\langle \varepsilon_i, \varepsilon_j^\vee \rangle = \delta_{ij}, \qquad \langle \varepsilon_i, K \rangle = 0,$$

$$\langle \delta, \varepsilon_j^\vee \rangle = 0, \qquad \langle \delta, K \rangle = 0$$

$$\langle \Lambda_0, \varepsilon_j^\vee \rangle = 0, \qquad \langle \Lambda_0, K \rangle = 1.$$

The group W^{ad} is the subgroup of $GL(\mathfrak{h}^*_{\mathbb{Z}})$ generated by $s_0, s_1, \ldots, s_{n-1}$. This action extends to an action of n-periodic permutations

$$W = \{ t_\mu v \mid \mu \in \mathbb{Z}^n, v \in S_n \}.$$

where, in the basis $\{\delta, \varepsilon_1, \ldots, \varepsilon_n, \Lambda_0\}$, the action is given by

$$t_\mu = \left(\begin{array}{c|ccc|c} 1 & -\mu_1 & \cdots & -\mu_n & -\frac{1}{2}|\mu|^2 \\ \hline & & & & \mu_1 \\ 0 & & 1 & & \vdots \\ & & & & \mu_n \\ \hline 0 & & 0 & & 1 \end{array}\right) \qquad v = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & v & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

for $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ and $v \in S_n$, where

$$-\tfrac{1}{2}|\mu|^2 = -\tfrac{1}{2}(\mu_1^2 + \cdots + \mu_n^2) \quad \text{and} \quad s_0 = t_{\varepsilon_1 - \varepsilon_n} s_1 \cdots s_{n-1} \cdots s_1$$

The group W acts on

$$\mathbb{C}[\mathfrak{h}_{\mathbb{Z}}^*] = \mathrm{span}\{x^\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*\}$$
$$= \mathbb{C}[q^{\pm 1}, x_1^{\pm 1}, \ldots, x_n^{\pm 1}, x^{\pm \Lambda_0}],$$

where $x^{a\delta + \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n + \ell\Lambda_0} = q^a x_1^{\mu_1} \cdots x_n^{\mu_n} x^{\ell\Lambda_0}$ and

$$w x^\lambda = x^{w\lambda} \qquad \text{for } \lambda \in \mathfrak{h}_{\mathbb{Z}}^* \text{ and } w \in W.$$

The <u>Demazure operators</u> are

$$D_\pi = t_{\varepsilon_1} s_1 \cdots s_{n-1} \quad \text{and}$$

$$D_{\pm\alpha_i} = (1 + s_i) \frac{1}{1 - x^{\pm\alpha_i}}$$

for $i \in \{0, 1, \ldots, n-1\}$.

<u>Remark</u> Let $D_i = D_{-\alpha_i}$ for $i \in \{0, 1, \ldots, n-1\}$.
For $w \in W^{ad}$ define

$$D_w = D_{i_1} \cdots D_{i_\ell} \quad \text{if} \quad w = s_{i_1} \cdots s_{i_\ell} \text{ is reduced.}$$

Let $\Lambda = a\delta + \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n + \ell \Lambda_0$ with
$\ell \in \mathbb{Z}_{\geq 0}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \lambda_1 - \ell$.
Let

$$L(\Lambda)_{\leq w} = U^+ v_{w\Lambda}, \quad \text{where}$$

v_Λ is a highest weight vector of weight Λ,
$v_{w\Lambda}$ is the w-extremal weight vector,
U^+ is the positive part of the quantum group.
Then

$$s_{\Lambda, w} = D_w X^\Lambda \text{ is the character of } L(\Lambda)_{\leq w}. //$$

Define <u>level ℓ Demazure operators</u> $D^{(\ell)}_{\pm\alpha_i}$ and $D^{(\ell)}_{\pi}$ on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, q^{\pm 1}]$ by

$$D^{(\ell)}_{\pi} = x^{-\ell\Lambda_0} D_{\pi} x^{\ell\Lambda_0}, \quad \text{and}$$

$$D^{(\ell)}_{\pm\alpha_i} = x^{-\ell\Lambda_0} D_{\pm\alpha_i} x^{\ell\Lambda_0} \quad \text{for } i \in \{0, 1, \ldots, n-1\}.$$

<u>Proposition</u> As operators on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, q^{\pm 1}]$

$$T_{\pi} = D^{(0)}_{\pi}, \qquad T_i = t^{\frac{1}{2}} D^{(0)}_{\alpha_i} + t^{-\frac{1}{2}}(D^{(0)}_{-\alpha_i} - 1)$$

for $i \in \{1, \ldots, n-1\}$

$$T_0 = t^{\frac{1}{2}} D^{(0)}_{\alpha_0} + t^{\frac{1}{2}}(D^{(0)}_{-\alpha_0} - 1) = T_{\pi} T_{n-1} T_{\pi}^{-1}$$

$$D^{(1)}_{\pi} = q^{-\frac{1}{2}} \tau^{\vee}_{\pi} y_n \quad \text{and}$$

$$y^{-\alpha_0^\vee} t^{\frac{1}{2}} \tau^{\vee}_0 = y_1 y_n^{-1} q t^{\frac{1}{2}} \tau^{\vee}_0$$

$$= D^{(1)}_{-\alpha_0} - t(D^{(1)}_{\alpha_0} - 1) + \frac{(1-t) y_1 y_n^{-1} q}{1 - y_1 y_n^{-1} q}$$

$$t^{\frac{1}{2}} \tau^{\vee}_i = D^{(1)}_{-\alpha_i} - t(D^{(1)}_{\alpha_i} - 1) + \frac{(1-t) y_i^{-1} y_{i+1}}{(1-t) y_i y_{i+1}}$$

for $i \in \{1, \ldots, n-1\}$.

Schubert polynomials for GL_∞

Define operators $\partial_i^{(\beta)}$ on $\mathbb{C}[[y_1, y_2, \ldots, h_1, h_2, \ldots ; \beta]]$ by

$$\partial_i^{(\beta)} = (1+s_i) \frac{1-\beta y_{i+1}}{y_i - y_{i+1}} \quad \text{for } i \in \mathbb{Z}_{>0}$$

For $w \in S_n$ define

$$\partial_w^{(\beta)} = \partial_{i_1}^{(\beta)} \cdots \partial_{i_\ell}^{(\beta)} \quad \text{if } w = s_{i_1} \cdots s_{i_\ell} \text{ is reduced.}$$

The double β-Schubert polynomials are given by

$$\mathfrak{G}_w(y, h; \beta) = \partial_{w^{-1}w_0}^{(\beta)} \Big(\prod_{i+j \leq n} (y_i + h_j + \beta y_i h_j) \Big)$$

where $w_0 \in S_n$ is given by $w_0(i) = n - i$.

Proposition Let z_i and x_i be the elements of $\mathbb{C}[[y_1, y_2, \ldots, h_1, h_2, \ldots ; \beta]]$ given by

$$z_i = \frac{1}{1-\beta h_i} \quad \text{and} \quad x_i = 1 - \beta y_i .$$

Then
$$\prod_{i+j \leq n} (y_i + h_j + \beta y_i h_j) = \beta^{-\frac{1}{2}n(n-1)} \prod_{i+j \leq n} (1 - x_i z_j^{-1})$$

and
$$\partial_i^{(\beta)} = \beta (1+s_i) \frac{1}{1 - x_i x_{i+1}^{-1}} = \beta D_{i,i+1}$$

So Macdonald polynomials specialized at $t=0$ might have something to do with Schubert polynomials.