2 Lecture 2, 2 March 2022: Macdonald polynomials

2.1 Page 1: Nonsymmetric Macdonald polynomials

Let $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let $\mathbb{Z}^n$ denote the set of length n sequences $\mu = (\mu_1, \ldots, \mu_n)$ of integers. The ring

$$\mathbb{C}[X] \text{ has basis } \quad \{x^\mu \mid \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n\}, \qquad \text{where} \quad x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}.$$

The symmetric group S_n acts on $\mathbb{C}[X]$ by permuting $x_1, \ldots, x_n$ so that

$$(s_i f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n).$$

The symmetric group S_n acts on $\mathbb{Z}^n$ by permuting the coordinates so that

$$s_i(\mu_1, \ldots, \mu_n) = (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \mu_i, \mu_{i+2}, \ldots, \mu_n).$$

Then $s_i x^\mu = x^{s_i\mu}$ for $\mu \in \mathbb{Z}^n$ and $i \in \{1, \ldots, n-1\}$.

For $f \in \mathbb{C}[X]$ and $i \in \{1, \ldots, n-1\}$ define

$$\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}}.$$

Define E_μ for $\mu \in \mathbb{Z}^n$ by the following recursive process:

(E0) $E_{(0,\ldots,0)} = 1$,

(E1) $E_{(\mu_n+1, \mu_1, \ldots, \mu_{n-1})} = q^{\mu_n} x_1 E_\mu(x_2, \ldots, x_n, q^{-1}x_1)$,

(E2) If $(\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ and $\mu_i > \mu_{i+1}$ then

$$E_{s_i\mu} = \Big(\partial_i x_i - t x_i \partial_i + \frac{(1-t)q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}}{1 - q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}}\Big) E_\mu,$$

where $v_\mu \in S_n$ is minimal length such that $v_\mu \mu$ is weakly increasing,

(E3) $E_{(\mu_1 - 1, \ldots, \mu_n - 1)} = x_1^{-1} \cdots x_n^{-1} E_{(\mu_1, \ldots, \mu_n)}$.

The set

$$\{E_\mu \mid \mu \in \mathbb{Z}^n\} \qquad \text{is a basis of } \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

2.2 Page 2: Symmetric Macdonald polynomials

Let

$$\mathbb{C}[X]^{S_n} = \{g \in \mathbb{C}[X] \mid \text{if } w \in S_n \text{ then } wg = g\}, \qquad \text{the ring of symmetric functions.}$$

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$, let $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ and let E_λ be the nonsymmetric Macdonald polynomial indexed by λ. The symmetric Macdonald polynomials P_λ are defined by

$$P_\lambda = P_\lambda(x; q, t) = \frac{1}{W_\lambda(t)} \sum_{w \in S_n} w\Big(E_\lambda \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j}\Big), \qquad \text{(Plambdadefn)}$$

where $W_\lambda(t)$ is the appropriate constant which makes the coefficient of x^λ equal to 1 in $P_\lambda(q,t)$. Various specializations of the $P_\lambda(x; q, t)$ have their own names.

$m_\lambda = P_\lambda(x; 0, 1)$	are the *monomial symmetric functions*,
$s_\lambda = P_\lambda(x; 0, 0)$	are the *Schur functions*,
$P_\lambda(x; 0, t)$	are the *Hall-Littlewood polynomials*.

Proposition 2.1. *If $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$ then $E_\lambda(0,t) = x^\lambda$.*

Using Proposition 2.1 gives formulas

$$m_\lambda = \sum_{\gamma \in S_n\lambda} x^\gamma, \qquad s_\lambda = \sum_{w \in S_n} w\Big(x^\lambda \prod_{i<j} \frac{x_i}{x_i - x_j}\Big) \tag{2.1}$$

$$\text{and} \qquad P_\lambda(0,t) = \frac{1}{W_\lambda(t)} \sum_{w \in S_n} w\Big(x^\lambda \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}\Big).$$

2.3 Page 3: Fermionic Macdonald polynomials

Let

$$(\mathbb{Z}^n)^+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \mid \lambda_1 \geq \cdots \geq \lambda_n\} \qquad \text{and}$$
$$(\mathbb{Z}^n)^{++} = \{\gamma = (\gamma_1, \ldots, \gamma_n) \mid \gamma_1 > \cdots > \gamma_n\}.$$

The map

$$\begin{matrix} (\mathbb{Z}^n)^+ & \to & (\mathbb{Z}^n)^{++} \\ \lambda & \mapsto & \lambda + \rho \end{matrix} \qquad \text{where} \quad \rho = (n-1, n-2, \ldots, 2, 1, 0), \tag{rhodefn}$$

is a bijection.

For $\lambda \in (\mathbb{Z}^n)^+$, the *fermionic Macdonald polynomial* $A_{\lambda+\rho}$ is

$$A_{\lambda+\rho}(q,t) = \Big(\prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}\Big) \sum_{w \in S_n} (-1)^{\ell(w)} w E_{\lambda+\rho}. \tag{Alambdadefn}$$

Theorem 2.2. *(Weyl character formula) Let $\lambda \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$. Then*

$$A_\rho = \prod_{i<j}(x_i - tx_j) \qquad \textit{and} \qquad P_\lambda(q, qt) = \frac{A_{\lambda+\rho}(q,t)}{A_\rho}.$$

2.4 Page 4: Eigenvalues

Define operators $T_1, \ldots, T_{n-1}$ and g on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by

$$T_i f = t^{-\frac{1}{2}}\big(t - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i)\big) f \qquad \text{and} \qquad (gf)(x_1, \ldots, x_n) = f(q^{-1}x_n, x_1, \ldots, x_{n-1}).$$

The *Cherednik-Dunkl operators* are

$$Y_1 = gT_{n-1}\cdots T_1, \quad Y_2 = T_1^{-1} Y_1 T_1^{-1}, \quad Y_3 = T_2^{-1} Y_1 T_2^{-1}, \quad \ldots, \quad Y_n = T_{n-1}^{-1} Y_{n-1} T_n^{-1}.$$

Theorem 2.3. *Let $\mu \in \mathbb{Z}^n$ and let $i \in \{1, \ldots, n\}$. Then*

$$Y_i E_\mu = q^{-\mu_i} t^{-(v_\mu(i)-1)+\frac{1}{2}(n-1)} E_\mu.$$

2.5 Page 5: Creation formulas

2.5.1 A creation formula for E_μ

The *intertwiners*, or *creation operators*, are

$$\tau_\pi^\vee = x_1 T_1 \cdots T_{n-1} \qquad \text{and} \qquad \tau_j^\vee = T_j + t^{-\frac{1}{2}} \frac{(1-t)Y_k}{Y_k - Y_j}, \qquad \text{(creationopsA)}$$

for $j \in \{1, \ldots, n-1\}$.

Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$. The minimal length permutation v_μ such that $v_\mu \mu$ is weakly increasing is given by

$$v_\mu(r) = 1 + \#\{r' \in \{1, \ldots, r-1\} \mid \mu_{r'} \leq \mu_r\} + \#\{r' \in \{r+1, \ldots, n\} \mid \mu_{r'} < \mu_r\},$$

for $r \in \{1, \ldots, n\}$. A *box in* μ is a pair (r, c) with $r \in \{1, \ldots, n\}$ and $c \in \{1, \ldots, \mu_r\}$. If $b = (r, c)$ is a box in μ then define

$$u_\mu(r, c) = \#\{r' \in \{1, \ldots, r-1\} \mid \mu_{r'} \leq c-1\} + \#\{r' \in \{r+1, \ldots, n\} \mid \mu_{r'} < c-1\}.$$

Theorem 2.4. *Let $\mu \in \mathbb{Z}_{\geq 0}^n$. Letting*

$$\tau_{u_\mu}^\vee = \prod_{(r,c) \in \mu} (\tau_{u_\mu(r,c)}^\vee \cdots \tau_2^\vee \tau_1^\vee \tau_\pi^\vee), \qquad \textit{then} \qquad E_\mu = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee 1,$$

where 1 *is the polynomial* $1 \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

2.5.2 Creation formulas for P_λ and $A_{\lambda+\delta}$

Let w_0 be the longest element of S_n so that

$$w_0(i) = n - i + 1, \text{ for } i \in \{1, \ldots, n\}, \qquad \text{and} \qquad \ell(w_0) = \frac{n(n-1)}{2} = \binom{n}{2}.$$

Let $z \in S_n$. A *reduced expression for* z is an expression for z as a product of s_i,

$$z = s_{i_1} \cdots s_{i_\ell}, \qquad \text{such that } i_1, \ldots, i_\ell \in \{1, \ldots, n-1\} \text{ and } \ell = \ell(z).$$

Define

$$T_z = T_{i_1} \cdots T_{i_\ell} \qquad \text{if } z = s_{i_1} \cdots s_{i_\ell} \text{ is a reduced word for } z.$$

The *bosonic symmetrizer* and the *fermionic symmetrizer* are

$$\mathbf{1}_0 = \sum_{z \in S_n} t^{\frac{1}{2}(\ell(z) - \ell(w_0))} T_z \qquad \text{and} \qquad \varepsilon_0 = \sum_{w \in S_n} (-t^{-\frac{1}{2}})^{\ell(z) - \ell(w_0)} T_z. \qquad \text{(symmdefsA)}$$

The creation formulas for P_λ and $A_{\lambda+\rho}$ are

2.6 Lecture 2: Notes and references

Macdonald [Mac, Ch. I (1.13)] uses the notation δ for $(n-1, \ldots, 2, 1, 0)$. The element ρ is such that $\langle \rho, \alpha_i^\vee \rangle = 1$ for the simple coroots $\alpha_i^\vee$. In the case of reductive groups which are not simple (like GL_n or Kac-Moody groups) the element ρ is not uniquely determined by these conditions. For the group GL_n, $\rho = (n-1, \ldots, 2, 1, 0)$ is a favorite choice, whereas for the group SL_n (which also has a type A root system) $\rho = (\frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-3}{2}, -\frac{n-1}{2})$ is the only option. Since this section is focused on Macdonald polynomials for type GL_n it seems sensible to use the notation ρ, in place of Macdonald's notation δ.