6 Lecture 6, 30 March 2022: Alcove walks, set valued tableaux and column strict tableaux

6.1 Page 1: Creation formulas

Define operators $T_1, \ldots, T_{n-1}$ and g on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by

$$T_i f = t^{-\frac{1}{2}}\big(t - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i)\big) f \qquad \text{and} \qquad (gf)(x_1, \ldots, x_n) = f(q^{-1}x_n, x_1, \ldots, x_{n-1}).$$

The *Cherdnik-Dunkl operators* are

$$Y_1 = gT_{n-1}\cdots T_1, \quad Y_2 = T_1^{-1}Y_1T_1^{-1}, \quad Y_3 = T_2^{-1}Y_1T_2^{-1}, \quad \ldots, \quad Y_n = T_{n-1}^{-1}Y_{n-1}T_n^{-1}.$$

The *intertwiners*, or *creation operators*, are

$$\begin{aligned} g^\vee &= x_1T_1\cdots T_{n-1} \qquad \text{and} \\ \tau_j^\vee &= T_j + f^+_{j+1,j} = T_j^{-1} + f^-_{j+1,j} \qquad \text{for } j \in \{1, \ldots, n-1\}, \end{aligned} \qquad \text{(creationops)}$$

where, for $k, j \in \mathbb{Z}$ with $j \neq k$,

$$f^-_{jk} = t^{-\frac{1}{2}}\frac{(1-t)Y_jY_k^{-1}}{1 - Y_jY_k^{-1}} = t^{-\frac{1}{2}}\frac{(1-t)Y_j}{Y_k - Y_j} \qquad \text{and} \qquad f^+_{jk} = t^{-\frac{1}{2}}\frac{(1-t)}{1 - Y_jY_k^{-1}} = t^{-\frac{1}{2}}\frac{(1-t)Y_k}{Y_k - Y_j}.$$

Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$. The minimal length permutation v_μ such that $v_\mu\mu$ is weakly increasing is given by

$$v_\mu(r) = 1 + \#\{r' \in \{1, \ldots, r-1\} \mid \mu_{r'} \leq \mu_r\} + \#\{r' \in \{r+1, \ldots, n\} \mid \mu_{r'} < \mu_r\},$$

for $r \in \{1, \ldots, n\}$. A *box in* μ is a pair (r, c) with $r \in \{1, \ldots, n\}$ and $c \in \{1, \ldots, \mu_r\}$. If $b = (r, c)$ is a box in μ then define

$$u_\mu(r, c) = \#\{r' \in \{1, \ldots, r-1\} \mid \mu_{r'} \leq c - 1\} + \#\{r' \in \{r+1, \ldots, n\} \mid \mu_{r'} < c - 1\}.$$

Define

$$\tau^\vee_{u_\mu} = \prod_{\text{boxes } (r,c) \text{ in } \mu} (\tau^\vee_{u_\mu(r,c)} \cdots \tau_2^\vee\tau_1^\vee g^\vee). \qquad \textbf{(thecreator)}$$

Let $z \in S_n$ and let $\mu \in \mathbb{Z}^n$. *The creation formula* for *the relative Macdonald polynomial* E^z_μ is

$$E^z_\mu = t^{-\frac{1}{2}(\ell(zv_\mu^{-1})}T_z\tau^\vee_{u_\mu}1, \qquad \textbf{(creationformula)}$$

where the action of T_w, X^μ, and Y_j on the polynomial 1 is given by

$$T_w1 = t^{\frac{1}{2}\ell(w)} \cdot 1, \qquad X^\mu \cdot 1 = x^\mu, \qquad Y_{i+\ell n} \cdot 1 = q^{-\ell}t^{-(v_\mu(i)-1)+\frac{1}{2}(n-1)} \cdot 1. \qquad \text{(actionon1)}$$

One can use the creation formula as the *definition* of the relative Macdonald polynomials. In terms of the relative Macdonald polynomials, the *nonsymmetric Macdonald polynomials* are

$$E_\mu = E^{\mathrm{id}}_\mu, \qquad \text{for } \mu \in \mathbb{Z}^n. \qquad \text{(nsaltdef)}$$

For $\lambda \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$ the *symmetric Macdonald polynomial* P_λ is

$$P_\lambda = \sum_{\mu \in S_n\lambda} E^{z_\mu}_\lambda, \qquad \text{(symmaltdef)}$$

where the sum is over all rearrangements of λ and $z_\mu \in S_n$ is the minimal length permutation such that $z_\mu\lambda = \mu$.

6.2 Page 2: Alcove walks formula

For $f(Y_1,\ldots,Y_n)\in\mathbb{C}(Y_1,\ldots,Y_n)$ define

$$(\pi f)(Y_1,\ldots,Y_n) = f(Y_2,\ldots,Y_n,Y_{n+1}) = f(Y_2,\ldots,Y_n,q^{-1}Y_1) \qquad \text{and}$$
$$(s_jf)(Y_1,\ldots,Y_n) = f(Y_1,\ldots,Y_{j-1},Y_{j+1},Y_j,Y_{j+1}\ldots,Y_n), \qquad \text{for } j\in\{1,\ldots,n-1\}.$$

The following relations will be proved in (taupastYrels1) and (taupastYrels2):

$$\tau_j^\vee f = (s_jf)\tau_j^\vee, \qquad \text{and} \qquad g^\vee f = (\pi f)g^\vee \tag{creationrel}$$

and

$$X^\mu T_{ws_i} = \begin{cases} X^\mu T_w T_i, & \text{if } \ell(ws_i)>\ell(w),\\ X^\mu T_w T_i^{-1}, & \text{if } \ell(ws_i)<\ell(w),\end{cases} \qquad \text{and} \qquad X^\mu T_w g^\vee = X^\mu X_{w(1)} T_{ws_1\cdots s_{n-1}}. \tag{Rmultrel}$$

These give that if $f\in\mathbb{C}(Y_1,\ldots,Y_n)$, $\mu\in\mathbb{Z}^n$ and $w\in S_n$ then

$$\begin{aligned} X^\mu T_w f(Y)\tau_j^\vee = X^\mu T_w \tau_j^\vee (s_jf)(Y) &= \begin{cases} X^\mu T_w (T_j^\vee + f^+_{-\alpha_j^\vee})(s_jf), & \text{if } \ell(ws_i)>\ell(w),\\ X^\mu T_w ((T_j^\vee)^{-1} + f^-_{-\alpha_j^\vee})(s_jf), & \text{if } \ell(ws_i)<\ell(w),\end{cases} \\ &= \begin{cases} X^\mu T_{ws_j}(s_jf) + X^\mu T_w (s_jf) f^+_{-\alpha_j^\vee}, & \text{if } \ell(ws_i)>\ell(w),\\ X^\mu T_{ws_j}(s_jf) + X^\mu T_w (s_jf) f^-_{-\alpha_j^\vee}, & \text{if } \ell(ws_i)<\ell(w),\end{cases}\end{aligned}$$

and

$$X^\mu T_w f g^\vee = X^\mu T_w g^\vee (\pi f) = X^\mu X_{w(1)} T_{ws_1\cdots s_{n-1}}(\pi f).$$

Inductively using these to compute $T_z\tau^\vee_{u_\mu} = T_z\tau^\vee_{i_1}\cdots\tau^\vee_{i_\ell} = (((T_z\tau^\vee_{i_1})\tau^\vee_{i_2})\cdots\tau^\vee_{i_{\ell-1}})\tau^\vee_{i_\ell}$ gives an expression for $T_z\tau^\vee_{u_\mu}$ with all the Xs on the left and all the Ys on the right,

$$T_z\tau^\vee_{u_\mu} = \sum_{F\subseteq\{1,\ldots,\ell\}} X^{\mathrm{end}^z(F)} T_{\varphi^z(F)} f_F^+ f_F^-, \tag{XleftYright}$$

where f_F^+ and f_F^- are products of f_{jk}^+ (respectively, f_{jk}^-) that fall out of the inductive process (we shall describe these explicitly in (**SVTformula**)). By applying (**XleftYright**) to the polynomial 1 and using the relations in (actionon1) gives

$$E_\mu^z = \sum_{F\subseteq\{1,\ldots,\ell\}} t^{\frac12(\ell(\varphi^z(F))-\ell(zv_\mu^{-1})}\mathrm{ev}_0^\rho(f_F^+f_F^-)x^{\mathrm{end}^z(F)}, \tag{alcovewalkformula}$$

where

$$\mathrm{ev}_0^\rho(f) = f(t^{\frac12(n-1)-0}, t^{\frac12(n-1)-1},\ldots,t^{\frac12(n-1)-(n-1)}) \qquad \text{for } f(Y_1,\ldots,Y_n)\in\mathbb{C}(Y_1,\ldots,Y_n).$$

The formula (**alcovewalkformula**) is the *alcove walk formula* for relative Macdonald polynomials. It provides an expansion of the relative Macdonald polynomials in terms of monomials. Modulo the proof of the relations (creationrel) and (Rmultrel) this page contains a complete proof of the alcove walk formula for relative Macdonald polynomials (and, by (nsaltdef) and (symmaltdef), also for the nonsymmetric Macdonald polynomials and the symmetric Macdonald polynomials).

6.3 Page 3: Set valued tableaux formula

This page provides the statement of a set-valued tableaux formula for relative Macdonald polynomials.

Use the notation γ_n for the n-cycle $\gamma_n = s_{n-1}\cdots s_1$ in S_n. For positive integers $k_1,\ldots,k_\ell$ such that $k_1+\cdots+k_\ell = n$ let

$$\gamma_{k_1}\times\cdots\times\gamma_{k_\ell} \qquad \text{be the disjoint product of cycles in} \qquad S_{k_1}\times\cdots\times S_{k_\ell}\subseteq S_n.$$

To give a formula for E^z_μ, fix $z\in S_n$ and $\mu=(\mu_1,\ldots,\mu_n)\in\mathbb{Z}^n_{\geq 0}$. Identify μ with *the set of boxes in* μ,

$$\mu = \{(r,c)\mid r\in\{1,\ldots,n\} \text{ and } c\in\{1,\ldots,\mu_r\}\}.$$

Order the boxes of μ by the values $r+nc$. The minimal length permutation v_μ such that $v_\mu\mu$ is weakly increasing is given by

$$v_\mu(r) = 1 + \#\{r'\in\{1,\ldots,r-1\}\mid \mu_{r'}\leq\mu_r\} + \#\{r'\in\{r+1,\ldots,n\}\mid \mu_{r'}<\mu_r\},$$

for $r\in\{1,\ldots,n\}$. For a box $(r,c)\in\mu$ and $i\in\{1,\ldots,u_\mu(r,c)\}$ define

$$\mathrm{sh}(i,r,c) = \mu_r - c + 1 = \mathrm{sh}(r,c), \qquad \text{and} \qquad \mathrm{ht}(i,r,c) = v_\mu(r) - i.$$

A *set valued tableau* T *of shape* μ is a choice of subset $T(r,c)\subseteq\{1,\ldots,u_\mu(r,c)\}$ for each box $(r,c)\in\mu$. More formally, a set valued tableau T of shape μ is a function

$$T\colon \mu\to\{\text{subsets of }\{1,\ldots,n\}\} \qquad \text{such that} \quad T(r,c)\subseteq\{1,\ldots,u_\mu(r,c)\}.$$

Let (r,c) be a box in μ. Let $z_{(r,c)} = z$ if (r,c) is the first box in μ and for a general box $(r,c)\in\mu$ define

$$z_{(r,c)} = z_{(r',c')}(\gamma_{u_{(r,c)}+1-\ell_p}\times\cdots\times\gamma_{\ell_2-\ell_1}\times\gamma_{\ell_1})\gamma_n^{-1},$$

where $T(r,c)=\{\ell_1,\ldots,\ell_p\}$ and $(r',c')\in\mu$ is the box before (r,c) in μ . Define

$$\begin{aligned} T^z_+(r,c) &= \{\ell_j\in T(r,c)\mid z_{(r,c)}(u_{(r,c)}+1-\ell_{j+1}) < z_{(r,c)}(u_{(r,c)}+1-\ell_j)\} \qquad \text{and} \\ T^z_-(r,c) &= \{\ell_j\in T(r,c)\mid z_{(r,c)}(u_{(r,c)}+1-\ell_{j+1}) > z_{(r,c)}(u_{(r,c)}+1-\ell_j)\}, \end{aligned}$$

where we make the conventions that $\ell_{p+1} = u_{(r,c)}+1$ and $z_{(r,c)}(0) = z_{(r,c)}(n)$. Define

$$\varphi^z(T) = z(b_{\max}) \text{ where } b_{\max} \text{ is the last box of } \mu, \qquad \#f(T) = \sum_{(r,c)\in\mu}\mathrm{Card}(T(r,c)),$$

and

$$\mathrm{cov}^z_\pm(T) = \Big(\sum_{(r,c)\in\mu}\sum_{i\in T^z_\mp(r,c)}\mathrm{ht}(i,r,c)\Big) \pm \tfrac{1}{2}\big(\ \ell(\varphi^z(T)) - \ell(zv_\mu^{-1}) - \#f(T)\ \big),$$

$$\mathrm{maj}^z_\pm(T) = \sum_{(r,c)}\sum_{i\in T^z_\mp(r,c)}\mathrm{sh}(i,r,c) = \sum_{(r,c)\in\mu}\mathrm{sh}(r,c)\cdot|T^z_\pm(r,c)| \qquad \text{and}$$

$$\mathrm{wt}^z_\pm(T) = q^{\pm\mathrm{cov}^z_\pm(T)}t^{\pm\mathrm{maj}^z_\pm(T)}\Big(\prod_{(r,c)\in\mu}\prod_{i\in T(r,c)}\frac{(1-t^{\pm 1})}{1-q^{\pm\mathrm{sh}_\mu(r,c,i)}t^{\pm\mathrm{ht}(r,c,i)}}\Big).$$

Then the relative Macdonald polynomial is

$$E^z_\mu = \sum_T \mathrm{wt}^z_+(T)x^T = \sum_T \mathrm{wt}^z_-(T)x^T, \qquad \text{where} \qquad x^T = \prod_{(r,c)\in\mu} x_{z_{(r,c)}(n)}, \qquad \textbf{(SVTformula)}$$

and the sum is over all set valued tableaux T of shape μ.

6.4 Page 4: Nonattacking fillings formula

Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ and $z \in S_n$. A *nonattacking filling* for (z, μ) is $T\colon \widehat{dg}(\mu) \to \{1, \ldots, n\}$ such that

(a) $T(i, 0) = z(i)$ for $i \in \{1, \ldots, n\}$ and

(b) if $b \in dg(\mu)$ and $a \in \mathrm{attack}_\mu(b)$ then $T(a) \neq T(b)$.

For example,

$$T = \begin{array}{c|ccccc} 1 & & & & & \\ 2 & 1 & 1 & 1 & 2 & \\ 3 & 3 & & & & \\ 4 & 4 & 4 & 5 & 4 & 4 \\ 5 & 5 & 2 & 3 & 3 & \end{array} \qquad \begin{array}{c} \text{is a nonattacking filling for } (z,\mu) \\ \\ \text{with } z = \mathrm{id} \in S_5 \text{ and } \mu = (0,4,1,5,4). \end{array} \tag{6.1}$$

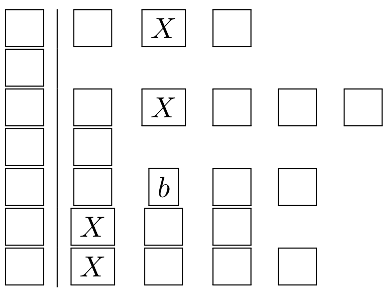
Let $\mu \in \mathbb{Z}^n_{\geq 0}$. Using cylindrical coordinates for boxes as specified in (1.9), define, for a box $b \in dg(\mu)$,

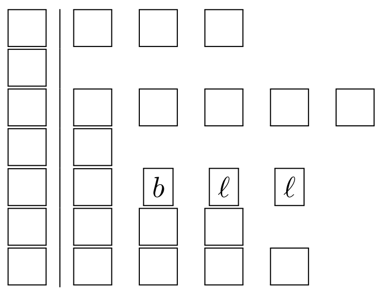
$$\mathrm{attack}_\mu(b) = \{b-1, \ldots, b-n+1\} \cap \widehat{dg}(\mu), \tag{6.2}$$

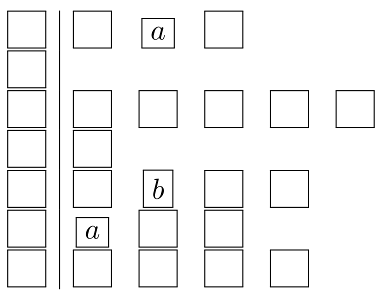
$$\mathrm{Nleg}_\mu(b) = (b + n\mathbb{Z}_{>0}) \cap dg(\mu) \quad \text{and} \tag{6.3}$$

$$\mathrm{Narm}_\mu(b) = \{a \in \mathrm{attack}_\mu(b) \mid \#\mathrm{Nleg}_\mu(a) \leq \#\mathrm{Nleg}_\mu(b)\}. \tag{6.4}$$

where $\#\mathrm{Nleg}_\mu(a)$ denotes the number of elements of $\mathrm{Nleg}_\mu(a)$. For example, with $\mu = (3,0,5,1,4,3,4)$ and $b = (5,2)$, which has cylindrical coordinate $b = 5 + 7 \cdot 2 = 19$ the sets $\mathrm{attack}_\mu(b)$, $\mathrm{Narm}_\mu(b)$ and $\mathrm{Nleg}_\mu(b)$ are pictured as

$\mathrm{attack}_\mu(b) =$ 

$\mathrm{Nleg}_\mu(b) =$ 

$\mathrm{Narm}_\mu(b) =$ 

Let $\mu \in \mathbb{Z}^n_{\geq 0}$ and $z \in S_n$ and let T be a nonattacking filling of shape (z, μ). For $b \in dg(\mu)$ let

$$\mathrm{bwn}_T(b) = \{a \in \mathrm{arm}_\mu(b) \mid T(b-n) > T(a) > T(b) \text{ or } T(b-n) < T(a) < T(b)\ \}. \tag{6.5}$$

The *weight of b in T* is

$$\mathrm{wt}_T(b) = \begin{cases} \Big(\dfrac{1-t}{1-q^{\#\mathrm{Nleg}_\mu(b)+1}t^{\#\mathrm{Narm}_\mu(b)+1}}\Big)t^{\#\mathrm{bwn}_T(b)}x_{T(b)}, & \text{if } T(b-n) > T(b), \\ \Big(\dfrac{(1-t)q^{\#\mathrm{Nleg}_\mu(b)+1}t^{\#\mathrm{Narm}_\mu(b)+1}}{1-q^{\#\mathrm{Nleg}_\mu(b)+1}t^{\#\mathrm{Narm}_\mu(b)+1}}\Big)t^{-1-\#\mathrm{bwn}_T(b)}x_{T(b)}, & \text{if } T(b-n) < T(b), \\ x_{T(b)}, & \text{if } T(b-n) = T(b), \end{cases} \tag{6.6}$$

and the *weight of T* is

$$\mathrm{wt}(T) = \prod_{b\in dg(\mu)} \mathrm{wt}_T(b), \qquad \text{a product over the boxes of } T. \tag{6.7}$$

The following theorem is the nonattacking fillings formula for relative Macdonald polynomials.

Theorem 6.1. *Let $\mu \in \mathbb{Z}^n_{\geq 0}$ and $z \in S_n$. Then the relative Macdonald polynomial E^z_μ is*

$$E^z_\mu = \sum_{T\in \mathrm{NAF}^z_\mu} \mathrm{wt}(T), \qquad \text{where the sum is over nonattacking fillings } T \text{ for } (z,\mu).$$

6.5 Page 5: Column strict tableaux formulas

Let

$$B(\lambda) = \{\text{column strict tableaux of shape } \lambda \text{ filled from } \{1,\ldots,n\}\}$$

For a column strict tableau T let $T(b)$ denote the entry in box b.

Let $T \in B(\lambda)$ and let $b \in \lambda$. Let $i \in \{1,\ldots,n\}$ with $i > T(b)$. Define

$$\begin{aligned} a(b,\leq i) &= \mathrm{Card}\{b' \in \mathrm{arm}_\lambda(b) \mid T(b') \leq i\}, & a(b,< i) &= \mathrm{Card}\{b' \in \mathrm{arm}_\lambda(b) \mid T(b') < i\}, \\ l(b,\leq i) &= \mathrm{Card}\{b' \in \mathrm{leg}_\lambda(b) \mid T(b') \leq i\}, & l(b,< i) &= \mathrm{Card}\{b' \in \mathrm{leg}_\lambda(b) \mid T(b') < i\} \end{aligned}$$

and

$$h_T(b,\leq i) = \frac{1-q^{a(b,\leq i)}t^{l(b,\leq i)+1}}{1-q^{a(b,\leq i)+1}t^{l(b,\leq i)}} \quad \text{and} \quad h_T(b,< i) = \frac{1-q^{a(b,< i)}t^{l(b,< i)+1}}{1-q^{a(b,< i)+1}t^{l(b,< i)}}. \tag{6.8}$$

Theorem 6.2. *For a column strict tableau $T \in B(\lambda)$ and a box $b \in \lambda$ define*

$$\psi_T = \prod_{b\in\lambda} \psi_T(b), \qquad \text{where} \qquad \psi_T(b) = \prod_{\substack{i>T(b), i\in T(\mathrm{arm}_\lambda(b)) \\ i\notin T(\mathrm{leg}\lambda(b))}} \frac{h_T(b,< i)}{h_T(b,\leq i)}$$

and $h_T(b,<i)$ and $h_T(b,\leq i)$ are as defined in (6.8). Then

$$P_\lambda = \sum_{T\in B(\lambda)} \psi_T x^T, \qquad \text{where} \quad x^T = x_1^{(\#1\text{s in } T)} \cdots x_n^{(\#n\text{s in } T)}.$$

6.6 Lecture 6: Notes and references

Theorem 6.2 follows [Mac, Ch. VI (7.13′)]. Theorem 6.1 summarizes [Al16, Def. 5 and Prop. 6] and [HHL06, Theorem 3.5.1]. Equation (**alcovewalkformula**) follows [RY08, Theorem 2.2 and Theorem 3.4].