10 Lecture 8, 13 April 2022: Orthogonality

10.1 Page 8.1: Definition of the inner product

Let $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Define an involution $\overline{}\colon \mathbb{C}[X] \to \mathbb{C}[X]$ by

$$\overline{f}(x_1, \ldots, x_n; q, t) = f(x_1^{-1}, \ldots, x_n^{-1}; q^{-1}, t^{-1}), \qquad \text{(keyinvdefn)}$$

Define

$$\nabla_{q,t} = \prod_{i \neq j} \frac{(x_i x_j^{-1}; q)_\infty}{(t x_i x_j^{-1}; q)_\infty} \qquad \text{and} \qquad \Delta_{q,t} = \nabla_{q,t} \prod_{i<j} \frac{1 - t x_i x_j^{-1}}{1 - x_i x_j^{-1}}. \qquad \text{(DnabladefnGL)}$$

Define a scalar product $(\ ,\ )_{q,t}\colon \mathbb{C}[X] \times \mathbb{C}[X] \to \mathbb{C}(q,t)$ by

$$(f_1, f_2)_{q,t} = \mathrm{ct}(f_1 \overline{f_2} \Delta_{q,t}), \qquad \text{where} \quad \mathrm{ct}(f) = (\text{constant term in } f), \qquad \text{(innproddefn)}$$

for $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

10.2 Page 8.2: Adjoints

Let y_n be the operator on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ given by

$$(y_n h)(x_1, \ldots, x_n) = h(x_1, \ldots, x_{n-1}, q^{-1} x_n).$$

The symmetric group S_n acts on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by permuting the variables $x_1, \ldots, x_n$. Define operators $T_1, \ldots, T_{n-1}$, T_π and $T_\pi^\vee$ on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by

$$T_i = t^{-\frac{1}{2}}\big(t - \frac{t x_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i)\big), \qquad T_\pi = s_1 s_2 \cdots s_{n-1} y_n, \qquad T_\pi^\vee = x_1 T_1 \cdots T_{n-1}, \qquad (10.1)$$

where $s_1, \ldots, s_{n-1}$ are the simple transpositions in S_n. The *Cherednik-Dunkl operators* are

$$Y_1 = T_\pi T_{n-1} \cdots T_1, \quad Y_2 = T_1^{-1} Y_1 T_1^{-1}, \quad Y_3 = T_2^{-1} Y_2 T_2^{-1}, \quad \ldots, \quad Y_n = T_{n-1}^{-1} Y_{n-1} T_n^{-1}. \qquad (10.2)$$

For a linear operator $M\colon \mathbb{C}[X] \to \mathbb{C}[X]$, the *adjoint of* M is the linear operator $M^*\colon \mathbb{C}[X] \to \mathbb{C}[X]$ determined by

$$(M f_1, f_2)_{q,t} = (f_1, M^* f_2)_{q,t}, \qquad \text{for } f_1, f_2 \in \mathbb{C}[X].$$

Proposition 10.1. *Let $i \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, n-1\}$. Then*

$$T_\pi^* = T_\pi^{-1}, \qquad x_i^* = x_i^{-1}, \qquad T_k^* = T_k^{-1}, \qquad Y_i^* = Y_i^{-1}, \qquad s_k^* = \frac{x_k - t x_{k+1}}{t x_k - x_{k+1}} s_k.$$

The *bosonic symmetrizer* and the *fermionic symmetrizer* are

$$\mathbf{1}_0 = \sum_{z \in S_n} t^{\frac{1}{2}(\ell(z) - \ell(w_0))} T_z \qquad \text{and} \qquad \varepsilon_0 = \sum_{z \in S_n} (-t^{-\frac{1}{2}})^{\ell(z) - \ell(w_0)} T_z.$$

Since $T_i^{-1} \mathbf{1}_0^* = T_i^* \mathbf{1}_0^* = (\mathbf{1}_0 T_i)^* = (t^{\frac{1}{2}} \mathbf{1}_0)^* = t^{-\frac{1}{2}} \mathbf{1}_0$ and

$$\mathbf{1}_0^* = T_{w_0}^{-1} + (\text{lower terms}) = T_{w_0} + (\text{lower terms}) \qquad \text{then} \qquad \mathbf{1}_0^* = \mathbf{1}_0. \qquad \text{(bosadjoint)}$$

Similarly, since $T_i^{-1} \varepsilon_0^* = T_i^* \varepsilon_0^* = (\varepsilon_0 T_i)^* = (-t^{-\frac{1}{2}} \varepsilon_0)^* = -t^{\frac{1}{2}} \varepsilon_0^*$ and

$$\varepsilon_0^* = T_{w_0}^{-1} + (\text{lower terms}) = T_{w_0} + (\text{lower terms}) \qquad \text{then} \qquad \varepsilon_0^* = \varepsilon_0. \qquad \text{(fermadjoint)}$$

10.2.1 Computation of the adjoints

Proposition 10.2. *Let $i \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, n-1\}$. Then*

$$g^* = g^{-1}, \qquad x_i^* = x_i^{-1}, \qquad T_k^* = T_k^{-1}, \qquad Y_i^* = Y_i^{-1}, \qquad s_k^* = \frac{x_k - tx_{k+1}}{tx_k - x_{k+1}} s_k.$$

Proof. Adjoint of multiplication by x_i:

$$(x_i f, g)_{q,t} = \mathrm{ct}(x_i f \overline{g} \Delta_{q,t}) = \mathrm{ct}(f \cdot \overline{x_i^{-1} g} \cdot \Delta_{q,t}) = (f, x_i^{-1} g)_{q,t}.$$

Adjoint of multiplication by π: Since

$$\pi \nabla_{q,t} = \pi \Big(\prod_{i \neq j} \frac{(x_i x_j^{-1}; q)_\infty}{(tx_i x_j^{-1}; q)_\infty} \Big) = \prod_{i \neq j} \frac{(x_{i+1} x_{j+1}^{-1}; q)_\infty}{(tx_{i+1} x_{j+1}^{-1}; q)_\infty}, \qquad \text{with } x_{n+1} = qx_1,$$

then

$$\begin{aligned} \pi \nabla_{q,t} &= \Big(\prod_{2 \leq i \neq j \leq n} \frac{(x_i x_j^{-1}; q)_\infty}{(tx_i x_j^{-1}; q)_\infty} \Big) \cdot \prod_{i=2}^{n} \frac{(q^{-1} x_i x_1^{-1}; q)_\infty (q x_1 x_i^{-1}; q)_\infty}{(q^{-1} t x_i x_1^{-1}; q)_\infty (q t x_1 x_i^{-1}; q)_\infty} \\ &= \Big(\prod_{1 \leq i \neq j \leq n} \frac{(x_i x_j^{-1}; q)_\infty}{(tx_i x_j^{-1}; q)_\infty} \Big) \cdot \prod_{i=2}^{n} \frac{(1 - q^{-1} x_i x_1^{-1})(1 - t x_1 x_i^{-1})}{(1 - q^{-1} t x_i x_1^{-1})(1 - x_1 x_i^{-1})} \end{aligned}$$

and

$$\pi \Big(\prod_{i<j} \frac{1 - t x_i x_j^{-1}}{1 - x_i x_j^{-1}} \Big) = \Big(\prod_{2 \leq i < j \leq n} \frac{1 - t x_i x_j^{-1}}{1 - x_i x_j^{-1}} \Big) \prod_{i=2}^{n} \frac{1 - q^{-1} t x_i x_1^{-1}}{1 - q^{-1} x_i x_1^{-1}}$$

so that $\pi \Delta_{q,t} = \Delta_{q,t}$. Thus

$$\begin{aligned} (\pi f_1, f_2)_{q,t} &= \mathrm{ct}\big((\pi f_1) \overline{f_2} \Delta_{q,t} \big) = \mathrm{ct}\big(\pi (f_1 \pi^{-1} (\overline{f_2} \Delta_{q,t})) \big) = \mathrm{ct}\big(f_1 \pi^{-1} (\overline{f_2} \Delta_{q,t}) \big) = \mathrm{ct}\big(f_1 \pi^{-1} (\overline{f_2} \Delta_{q,t}) \big) \\ &= \mathrm{ct}\big(f_1 \pi^{-1} (\overline{f_2}) \Delta_{q,t} \big) = \mathrm{ct}\big(f_1 \cdot \overline{\pi^{-1} f_2} \cdot \Delta_{q,t} \big) = (f_1, \pi^{-1} f_2)_{q,t}. \end{aligned}$$

Let

$$c_{ij} = t^{-\frac{1}{2}} \frac{1 - t x_i x_j^{-1}}{1 - x_i x_j^{-1}} \quad \text{so that} \quad \overline{c_{ij}} = t^{\frac{1}{2}} \frac{1 - t^{-1} x_i^{-1} x_j}{1 - x_i^{-1} x_j} = c_{ij}$$

and

$$s_k \Big(\prod_{i<j} \frac{1 - t x_i x_j^{-1}}{1 - x_i x_j^{-1}} \Big) = \Big(\prod_{i<j} \frac{1 - t x_i x_j^{-1}}{1 - x_i x_j^{-1}} \Big) \Big(\frac{1 - t x_{k+1} x_k^{-1}}{1 - x_{k+1} x_k^{-1}} \Big) \Big(\frac{1 - x_k x_{k+1}^{-1}}{1 - t x_k x_{k+1}^{-1}} \Big) = \Big(\prod_{i<j} \frac{1 - t x_i x_j^{-1}}{1 - x_i x_j^{-1}} \Big) \frac{c_{k+1,k}}{c_{k,k+1}}.$$

Then

$$\begin{aligned} (s_k f_1, f_2)_{q,t} &= \mathrm{ct}\big((s_k f_1) \overline{f_2} \Delta_{q,t} \big) = \mathrm{ct}\big(s_k (f_1 (s_k (\overline{f_2} \Delta_{q,t}))) \big) = \mathrm{ct}\big(f_1 (s_k (f_2 \Delta_{q,t})) \big) \\ &= \mathrm{ct}\Big(f_1 (s_k \overline{f_2}) \Delta_{q,t} \frac{c_{k+1,k}}{c_{k,k+1}} \Big) = \mathrm{ct}\Big(f_1 \overline{\frac{c_{k+1,k}}{c_{k,k+1}} (s_k f_2)} \Delta_{q,t} \Big) = (f_1, \frac{c_{k+1,k}}{c_{k,k+1}} (s_k f_2))_{q,t}, \end{aligned}$$

where

$$\frac{c_{k+1,k}}{c_{k,k+1}} = \frac{\frac{1 - t x_{k+1} x_k^{-1}}{1 - x_{k+1} x_k^{-1}}}{\frac{1 - t x_k x_{k+1}^{-1}}{1 - x_k x_{k+1}^{-1}}} = \frac{\frac{x_k - t x_{k+1}}{x_k - x_{k+1}}}{\frac{x_{k+1} - t x_k}{x_{k+1} - x_k}} = \frac{x_k - t x_{k+1}}{t x_k - x_{k+1}}.$$

If $i \in \{1, \ldots, n-1\}$ then

$$\begin{aligned} T_i^* &= \big(t^{\frac{1}{2}} + c_{i,i+1}(x)(s_i - 1)\big)^* = t^{-\frac{1}{2}} + (s_i^* - 1)(c_{i,i+1}(x))^* \\ &= t^{-\frac{1}{2}} + \Big(\frac{c_{i,i+1}(x)}{c_{i+1,i}(x)} s_i - 1\Big) c_{i,i+1}(x) = t^{-\frac{1}{2}} + c - i, i+1(x)(s_i - 1) = T_{s_i}^{-1}. \end{aligned}$$

Then

$$Y_1^* = (T_\pi T_{n-1} \cdots T_1)^* = T_1^{-1} \cdots T_{n-1}^{-1} T_\pi^{-1} = (T_\pi T_{n-1} \cdots T_1)^{-1} = Y_1^{-1},$$

and if $j \in \{2, \ldots, n\}$ then

$$Y_j^* = (T_{j-1}^{-1} Y_{j-1} T_{j-1}^{-1})^* = T_{j-1} Y_{j-1}^{-1} T_{j-1} = (T_{j-1}^{-1} Y_{j-1} T_{j-1}^{-1})^{-1} = Y_j^{-1}.$$

□

10.3 Page 8.3: Orthogonality

For $\mu \in \mathbb{Z}^n$ the *electronic Macdonald polynomial* E_μ is the (unique) element $E_\mu \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ such that

$$Y_i E_\mu = q^{-\mu_i} t^{-(v_\mu(i)-1)+\frac{1}{2}(n-1)} E_\mu, \qquad \text{and the coefficient of } x_1^{\mu_1} \cdots x_n^{\mu_n} \text{ in } E_\mu \text{ is } 1, \tag{10.3}$$

where $v_\mu \in S_n$ is the minimal length permutation such that $v_\mu \mu$ is weakly increasing.
Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n$.

$$\text{The } \textit{bosonic Macdonald polynomial } P_\lambda \text{ is} \qquad P_\lambda = \sum_{\nu \in S_n \lambda} t^{\frac{1}{2}\ell(z_\nu)} T_{z_\nu} E_\lambda, \tag{10.4}$$

where the sum is over rearrangements ν of λ and $z_\nu \in S_n$ is minimal length such that $\nu = z_\nu \lambda$.
Let $\rho = (n-1, n-2, \ldots, 2, 1, 0)$. The *fermionic Macdonald polynomial* $A_{\lambda+\rho}$ is

$$A_{\lambda+\rho} = (-t)^{\ell(w_0)} \sum_{z \in S_n(\lambda+\rho)} (-t^{-\frac{1}{2}})^{\ell(z)} T_z E_{\lambda+\rho}. \tag{10.5}$$

The relations $Y_i^* = Y_i^{-1}$ in combination with the knowledge of the eigenvalues for the action of the Y_i on the E_μ gives the following orthogonality relations for Macdonald poylnomials.

Proposition 10.3.

(a) Let $\lambda, \mu \in \mathbb{Z}^n$. If $\mu \neq \lambda$ then $(E_\lambda, E_\mu)_{q,t} = 0$.

(b) Let $\lambda, \mu \in (\mathbb{Z}^n)^+$. If $\mu \neq \lambda$ then $(P_\lambda, P_\mu)_{q,t} = 0$.

(b) Let $\lambda, \mu \in (\mathbb{Z}^n)^+$. If $\mu \neq \lambda$ then $(A_{\lambda+\delta}, A_{\mu+\delta})_{q,t} = 0$.

10.3.1 Proof of the orthogonality relations

Proposition 10.4.

(a) Let $\lambda, \mu \in \mathbb{Z}^n$. If $\mu \neq \lambda$ then $(E_\lambda, E_\mu)_{q,t} = 0$.

(b) Let $\lambda, \mu \in (\mathbb{Z}^n)^+$. If $\mu \neq \lambda$ then $(P_\lambda, P_\mu)_{q,t} = 0$.

(b) Let $\lambda, \mu \in (\mathbb{Z}^n)^+$. If $\mu \neq \lambda$ then $(A_{\lambda+\delta}, A_{\mu+\delta})_{q,t} = 0$.

Proof. Let $i \in \{1, \ldots, n\}$. Then, by Theorem 2.5,

$$q^{-\lambda_i} t^{-(v_\lambda(i)-1)+\frac{1}{2}(n-1)} (E_\lambda, E_\mu)_{q,t} = (Y_i E_\lambda, E_\mu)_{q,t} = (E_\lambda, Y_i^{-1} E_\mu)_{q,t} = (E_\lambda, q^{\mu_i} t^{(v_\mu(i)-1)\frac{1}{2}(n-1)} E_\mu)_{q,t}$$
$$= \overline{q^{\mu_i} t^{(v_\mu(i)-1)-\frac{1}{2}(n-1)}} (E_\lambda, E_\mu)_{q,t} = q^{-\mu_i} t^{-(v_\mu(i)-1)+\frac{1}{2}(n-1)} (E_\lambda, E_\mu)_{q,t}.$$

If $(E_\lambda, E_\mu)_{q,t} \neq 0$ then $q^{-\lambda_i} = q^{-\mu_i}$ for $i \in \{1, \ldots, n\}$. Thus $\lambda_i = \mu_i$ for $i \in \{1, \ldots, n\}$ and so $\lambda = \mu$ (and $v_\lambda = v_\mu$).

Parts (b) and (c) follow from (a) and the E-expansions in Proposition 4.6. □

10.4 Page 8.4: Reductions for norms

For $\mu \in \mathbb{Z}^n$ let u_μ and t_μ be the n-periodic permutations given by

$$t_\mu(i) = i + n\mu_i \qquad \text{and} \qquad u_\mu = t_\mu v_\mu^{-1},$$

where $v_\mu \in S_n$ is minimal length such that $v_\mu \mu$ is weakly decreasing. For $i, j \in \{1, \ldots, n\}$ and $\ell \in \mathbb{Z}$ define

$$c_{(i,j+\ell n)}(Y) = \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} q^\ell Y_i Y_j^{-1}}{1 - q^\ell Y_i Y_j^{-1}}.$$

For an n-periodic permutation w define

$$\mathrm{Inv(w)} = \left\{ (i,k) \mid \begin{array}{l} i \in \{1, \ldots, n\}, k \in \mathbb{Z} \\ i < k \text{ and } w(i) > w(k) \end{array} \right\} \qquad \text{and} \qquad c_w(Y) = \prod_{(i,k) \in \mathrm{Inv}(w)} c_{(i,k)}(Y).$$

Let $\mathrm{ev}_\mu^t \colon \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}] \to \mathbb{C}$ be the homormophism given by

$$\mathrm{ev}_\mu^t(Y_i) = q^{-\mu_i} t^{-(v_\mu(i)-1)+\frac{1}{2}(n-i)}.$$

Proposition 10.5. *Let $\mu, \lambda \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$. Then*

$$(E_\mu, E_\mu)_{q,t} = \mathrm{ev}_0^t(c_{u_\mu}(Y) c_{u_\mu}(Y^{-1})) \cdot (1,1)_{q,t} \ ,$$

$$(P_\lambda, P_\lambda)_{q,t} = \frac{W_0(t)}{W_\lambda(t)} \mathrm{ev}_\lambda^t(c_{v_\lambda}(Y^{-1})) \cdot (E_\lambda, E_\lambda)_{q,t} \ ,$$

$$(A_{\lambda+\rho}, A_{\lambda+\rho})_{q,t} = W_0(t) \mathrm{ev}_{\lambda+\rho}^t(c_{w_0}(Y)) \cdot (E_{\lambda+\rho}, E_{\lambda+\rho})_{q,t} \ .$$

Alternatively,

$$(E_\mu, E_\mu)_{q,t} = \Big(\prod_{(r,c) \in \mu} \prod_{i=1}^{u_\mu(r,c)} \frac{(1 - q^{\mu_r - c+1} t^{v_\mu(r)-i+1})(1 - q^{\mu_r - c+1} t^{v_\mu(r)-i-1})}{(1 - q^{\mu_r - c+1} t^{v_\mu(r)-i})^2} \Big) \cdot (1,1)_{q,t},$$

$$(P_\lambda, P_\lambda)_{q,t} = \frac{W_0(t)}{W_\lambda(t)} \Big(\prod_{i<j} \frac{1 - q^{\lambda_i - \lambda_j} t^{j-i-1}}{1 - q^{\lambda_i - \lambda_j} t^{j-i}} \Big) \cdot (E_\lambda, E_\lambda)_{q,t}.$$

$$(A_{\lambda+\rho}, A_{\lambda+\rho})_{q,t} = W_0(t^{-1}) \Big(\prod_{i<j} \frac{1 - q^{\lambda_i - \lambda_j + j - i} t^{j-i+1}}{1 - q^{\lambda_i - \lambda_j + j - i} t^{j-i}} \Big) \cdot (E_{\lambda+\rho}, E_{\lambda+\rho})_{q,t}.$$

Proof. First note that

$$(\tau_i^\vee)^* = \Big(T_i + \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}{1 - Y_i^{-1}Y_{i+1}}\Big)^* = T_i^* + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - Y_iY_{i+1}^{-1}} = T_i^{-1} + \frac{(t^{-\frac{1}{2}} - t^{\frac{1}{2}})Y_i^{-1}Y_{i+1}}{1 - Y_i^{-1}Y_{i+1}} = \tau_i^\vee,$$

and

$$(\tau_i^\vee)^2 = \frac{(1 - tY_iY_{i+1}^{-1})(1 - tY_i^{-1}Y_{i+1})}{(1 - Y_iY_{i+1}^{-1})(1 - Y^{-1}Y_{i+1})} = c_{i,i+1}(Y)c_{i,i+1}(Y^{-1}).$$

Then, using the creation formula for E_μ (Theorem 2.6),

$$\begin{aligned}(E_\mu, E_\mu)_{q,t} &= (t^{-\frac{1}{2}\ell(v_\mu^{-1})}\tau_{u_\mu}^\vee \mathbf{1}_Y, t^{-\frac{1}{2}\ell(v_\mu^{-1})}\tau_{u_\mu}^\vee \mathbf{1}_Y)_{q,t} = (\tau_{u_\mu^{-1}}^\vee \tau_{u_\mu}^\vee \mathbf{1}_Y, \mathbf{1}_Y)_{q,t} \\ &= (c_{u_\mu}(Y)c_{u_\mu}(Y^{-1})\mathbf{1}_Y, \mathbf{1}_Y)_{q,t} = \mathrm{ev}_0^t(c_{u_\mu}(Y)c_{u_\mu}(Y^{-1})) \cdot (1,1)_{q,t}.\end{aligned}$$

(b) Recall from (symmprops) that

$$\mathbf{1}_0^2 = t^{-\frac{1}{2}\ell(w_0)}W_0(t)\mathbf{1}_0 \qquad \text{and} \qquad \varepsilon_0^2 = (-1)^{\ell(w_0)}t^{-\frac{1}{2}\ell(w_0)}W_0(t)\varepsilon_0.$$

Recall from Proposition 4.6 that

$$\begin{aligned} P_\lambda &= \sum_{\mu \in W_0\lambda} b_\lambda^\mu E_\mu, && \text{with} && b_\lambda^\lambda = \mathrm{ev}_\lambda^\rho(c_{v_\lambda}(Y)), \\ A_{\lambda+\rho} &= \sum_{\mu \in W_0(\lambda+\rho)} d_{\lambda+\rho}^\mu E_\mu, && \text{with} && d_{\lambda+\rho}^{\lambda+\rho} = \mathrm{ev}_{\lambda+\rho}^\rho(c_{w_0}(Y^{-1})).\end{aligned}$$

Since $W_\lambda(t^{-1}) = t^{-\ell(w_\lambda)}W_\lambda(t)$ and $\ell(w_0) - \ell(w_\lambda) = \ell(v_\lambda)$ then using $P_\lambda = \frac{t^{\frac{1}{2}\ell(w_0)}}{W_\lambda(t)}\mathbf{1}_0E_\lambda$ gives

$$\begin{aligned}(P_\lambda, P_\lambda)_{q,t} &= \Big(\frac{t^{\frac{1}{2}\ell(w_0)}}{W_\lambda(t)}\mathbf{1}_0E_\lambda, \frac{t^{\frac{1}{2}\ell(w_0)}}{W_\lambda(t)}\mathbf{1}_0E_\lambda\Big)_{q,t} = \frac{1}{W_\lambda(t)W_\lambda(t^{-1})}(\mathbf{1}_0^2E_\lambda, E_\lambda)_{q,t} \\ &= \frac{t^{-\frac{1}{2}\ell(w_0)}W_0(t)}{W_\lambda(t)W_\lambda(t^{-1})}(\mathbf{1}_0E_\lambda, E_\lambda)_{q,t} = \frac{t^{-\ell(w_0)}W_0(t)}{W_\lambda(t^{-1})}(P_\lambda, E_\lambda)_{q,t} \\ &= \frac{t^{-\ell(w_0)}W_0(t)}{t^{-\ell(w_\lambda)}W_\lambda(t)}b_\lambda^\lambda(E_\lambda, E_\lambda)_{q,t} = \frac{t^{-\ell(v_\lambda)}W_0(t)}{W_\lambda(t)}b_\lambda^\lambda(E_\lambda, E_\lambda)_{q,t}\end{aligned}$$

Similarly, using $A_{\lambda+\rho} = t^{\frac{1}{2}\ell(w_0)}\varepsilon_0E_{\lambda+\rho}$ gives

$$\begin{aligned}(A_{\lambda+\rho}, A_{\lambda+\rho})_{q,t} &= \big(t^{\frac{1}{2}\ell(w_0)}\varepsilon_0E_{\lambda+\rho},\ t^{\frac{1}{2}\ell(w_0)}\varepsilon_0E_{\lambda+\rho}\big)_{q,t} = \big(\varepsilon_0^2E_{\lambda+\rho},\ E_{\lambda+\rho}\big)_{q,t} \\ &= (-1)^{\ell(w_0)}t^{-\frac{1}{2}\ell(w_0)}W_0(t)\big(\varepsilon_0E_{\lambda+\rho},\ E_{\lambda+\rho}\big)_{q,t} = (-1)^{\ell(w_0)}t^{-\ell(w_0)}W_0(t)\big(A_{\lambda+\rho},\ E_{\lambda+\rho}\big)_{q,t} \\ &= (-1)^{\ell(w_0)}W_0(t^{-1})d_{\lambda+\rho}^{\lambda+\rho}\big(E_{\lambda+\rho},\ E_{\lambda+\rho}\big)_{q,t}.\end{aligned}$$

By Proposition 4.6,

$$b_\lambda^\lambda = \prod_{\substack{1\le i<j\le n \\ \lambda_i > \lambda_j}} t\Big(\frac{1 - q^{\lambda_i-\lambda_j}t^{v_\lambda(i)-v_\lambda(j)-1}}{1 - q^{\lambda_i-\lambda_j}t^{v_\lambda(i)-v_\lambda(j)}}\Big) = t^{\ell(v_\lambda)} \prod_{\substack{1\le i<j\le n \\ \lambda_i > \lambda_j}} \Big(\frac{1 - q^{\lambda_i-\lambda_j}t^{v_\lambda(i)-v_\lambda(j)-1}}{1 - q^{\lambda_i-\lambda_j}t^{v_\lambda(i)-v_\lambda(j)}}\Big)$$

and, since $v_{\lambda+\rho}(i) = n - i$ then

$$d^{\lambda+\rho}_{\lambda+\rho} = \prod_{\substack{1\le i<j\le n\\ (\lambda+\rho)_i>(\lambda+\rho)_j}} (-1)\Big(\frac{1-q^{(\lambda+\rho)_i-(\lambda+\rho)_j}t^{v_{\lambda+\rho}(i)-v_{\lambda+\rho}(j)+1}}{1-q^{(\lambda_\rho)_i-(\lambda+\rho)_j}t^{v_{\lambda+\rho}(i)-v_{\lambda+\rho}(j)}}\Big)$$

$$= \prod_{1\le i<j\le n} (-1)\Big(\frac{1-q^{\lambda_i-\lambda_j+j-i}t^{j-i+1}}{1-q^{\lambda_i-\lambda_j+j-i}t^{j-i}}\Big) = (-1)^{\ell(w_0)} \prod_{1\le i<j\le n} \Big(\frac{1-q^{\lambda_i-\lambda_j+j-i}t^{j-i+1}}{1-q^{\lambda_i-\lambda_j+j-i}t^{j-i}}\Big)$$

□

10.5 Page 5: Formulas for norms and the constant term

Proposition 10.6. *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \ge \cdots \ge \lambda_n$. Then*

$$(P_\lambda(q,qt), P_\lambda(q,qt))_{q,qt} = \frac{W_0(qt)}{W_0(t^{-1})}\mathrm{ev}^t_{\lambda+\rho}\Big(\frac{c_{w_0}(Y^{-1})}{c_{w_0}(Y)}\Big)(P_{\lambda+\rho}(q,t), P_{\lambda+\rho}(q,t))_{q,t},$$

Proof. Using Proposition 7.5 and the Weyl character formula Theorem 7.6,

$$(P_\lambda(q,qt), P_\lambda(q,qt))_{q,qt} = \frac{W_0(qt)}{W_0(t^{-1})}(A_\rho(t)P_\lambda(q,qt), A_\rho(t)P_\lambda(q,qt))_{q,t}$$

$$= \frac{W_0(qt)}{W_0(t^{-1})}(A_{\lambda+\rho}(q,t), A_{\lambda+\rho}(q,t))_{q,t} = \frac{W_0(qt)}{W_0(t^{-1})}\mathrm{ev}^t_{\lambda+\rho}\Big(\frac{c_{w_0}(Y^{-1})}{c_{w_0}(Y)}\Big)(P_{\lambda+\rho}(q,t), P_{\lambda+\rho}(q,t))_{q,t},$$

since, by Proposition 10.5 (see also [Mac03, (5.7.12)]),

$$\frac{(A_{\lambda+\rho}, A_{\lambda+\rho})_{q,t}}{(P_{\lambda+\rho}, P_{\lambda+\rho})_{q,t}} = \mathrm{ev}^t_{\lambda+\rho}\Big(\frac{c_{w_0}(Y^{-1})}{c_{w_0}(Y)}\Big).$$

□

Theorem 10.7. *Let $\lambda \in \mathbb{Z}^n$ with $\lambda_1 \ge \cdots \ge \lambda_n$. Let $k \in \mathbb{Z}_{>0}$. Then*

$$\langle P_\lambda(q,q^k), P_\lambda(q,q^k)\rangle_{q,q^k} = \prod_{i<j}\prod_{r=1}^{k-1} \frac{1-q^{\lambda_i-\lambda_j+r}t^{j-i}}{1-q^{\lambda_i-\lambda_j-r}t^{j-i}}.$$

Proof. Note that

$$c_{\alpha^\vee}(Y^{-1},t) = \frac{t^{-\frac12}-t^{\frac12}Y^{-\alpha^\vee}}{1-Y^{-\alpha^\vee}} = \frac{t^{\frac12}-t^{-\frac12}Y^{\alpha^\vee}}{1-Y^{\alpha^\vee}} = t^{\frac12}\frac{1-t^{-1}Y^{\alpha^\vee}}{1-Y^{\alpha^\vee}} = t^{\frac12}c_{\alpha^\vee}(Y,t^{-1}).$$

Assume $t = q^k$. Let R_k be the right hand side of the statement,

$$R_{\lambda,k} = \prod_{i<j}\prod_{r=1}^{k-1} \frac{1-q^{\lambda_i-\lambda_j+r}t^{j-i}}{1-q^{\lambda_i-\lambda_j-r}t^{j-i}}.$$

Then

$$\frac{R_{\lambda,k+1}}{R_{\lambda+\rho,k}} = \prod_{i<j}\Big(\prod_{r=1}^{k}\frac{1-q^{\lambda_i-\lambda_j+r}q^{(k+1)(j-i)}}{1-q^{\lambda_i-\lambda_j-r}q^{(k+1)(j-i)}}\Big)\cdot\Big(\prod_{r=1}^{k-1}\frac{1-q^{\lambda_i-\lambda_j+j-i-r}q^{k(j-i)}}{1-q^{\lambda_i-\lambda_j+j-i+r}q^{k(j-i)}}\Big)$$

$$= \prod_{i<j}\frac{1-q^{\lambda_i-\lambda_j+k}q^{(k+1)(j-i)}}{1-q^{\lambda_i-\lambda_j-k}q^{(k+1)(j-i)}} = \prod_{i<j}\frac{1-q^{\lambda_i-\lambda_j+(k+1)(j-i)}q^k}{1-q^{\lambda_i-\lambda_j+(k+1)(j-i)}q^{-k}}$$

$$= \prod_{\alpha\in R^+}\frac{1-q^{\langle\lambda+(k+1)\rho,\alpha\rangle}q^k}{1-q^{\langle\lambda+(k+1)\rho,\alpha\rangle}q^{-k}} = \mathrm{ev}_{\lambda+(k+1)\rho}\Big(\frac{c_{w_0}(Y,q^k)}{c_{w_0}(Y,(q^k)^{-1})}\Big) = \frac{\langle P_\lambda(q,qq^k), P_\lambda(q,qq^k)\rangle_{q,qq^k}}{\langle P_{\lambda+\rho}(q,q^k), P_{\lambda+\rho}(q,q^k)\rangle_{q,q^k}},$$

where the last equality follows from Proposition 10.6 and Proposition 10.10 SYMMCOMP. The result then follows by induction since the base case is

$$\langle P_\lambda(q,q), P_\lambda(q,q)\rangle_{q,q} = \langle s_\lambda, s_\lambda\rangle = 1 = R_1.$$

□

Proposition 10.8. *Let $k \in \mathbb{Z}_{>0}$. Then*

$$\langle 1,1\rangle_{q,q^k} = \prod_{h=2}^{n-1} \begin{bmatrix} hk-1 \\ k-1 \end{bmatrix} \qquad \text{and} \qquad (1,1)_{q,q^k} = \prod_{i=2}^{n} \begin{bmatrix} ik \\ k \end{bmatrix}.$$

Proof. Using that $1 = P_0(q,q^k)$ then Theorem 10.7 gives

$$\begin{aligned}
\langle 1,1\rangle_{q,q^k} &= \prod_{i<j}\prod_{r=1}^{k-1} \frac{1-q^r t^{j-i}}{1-q^{-r}t^{j-i}} = \prod_{i<j}\prod_{r=1}^{k-1} \frac{1-q^{k(j-i)+r}}{1-q^{k(j-i)-r}} \\
&= \prod_{h=1}^{n-1} \prod_{\substack{i<j \\ j-i=h}} \frac{(1-q^{kh+1})\cdots(1-q^{kh+k-1})}{(1-q^{kh-1})\cdots(1-q^{kh-(k-1)})} = \prod_{h=1}^{n-1} \frac{\big((1-q^{kh+1})\cdots(1-q^{(k+1)h-1})\big)^{n-h}}{\big((1-q^{(k-1)h+1})\cdots(1-q^{kh-1})\big)^{n-h}} \\
&= \Big(\frac{1}{(1-q^{(k-1)+1})\cdots(1-q^{k-1})}\Big)^{n-1} \prod_{h=2}^{n} (1-q^{k(h-1)+1})\cdots(1-q^{kh-1}) = \prod_{h=2}^{n-1} \begin{bmatrix} hk-1 \\ k-1 \end{bmatrix}.
\end{aligned}$$

Letting $t = q^k$,

$$(1,1)_{q,t} = W_0(t)\langle 1,1\rangle_{q,t} = \Big(\prod_{i=2}^{n} \frac{1-t^i}{1-t}\Big) \prod_{i=2}^{n} \begin{bmatrix} ik-1 \\ k-1 \end{bmatrix} = \Big(\prod_{i=2}^{n} \frac{1-q^{ik}}{1-q^k}\Big) \prod_{i=2}^{n} \begin{bmatrix} ik-1 \\ k-1 \end{bmatrix} = \prod_{i=2}^{n} \begin{bmatrix} ik \\ k \end{bmatrix}.$$

□

Remark 10.9. Converting to general q and t. Let

$$\Delta^+(t) = \prod_{1\le i<j\le n} \frac{(Y_iY_j^{-1};q)_\infty}{(tY_iY_j^{-1};q)_\infty} \qquad \text{and} \qquad \Delta^-(t^{-1}) = \prod_{1\le i<j\le n} \frac{(qY_i^{-1}Y_j;q)_\infty}{(t^{-1}qY_i^{-1}Y_j;q)_\infty}.$$

If $k \in \mathbb{Z}_{>0}$ and $t = q^k$ then

$$\begin{aligned}
\langle P_\lambda, P_\lambda\rangle_{q,t} &= \prod_{i<j}\prod_{r=1}^{k-1} \frac{1-q^{\lambda_i-\lambda_j+r}t^{j-i}}{1-q^{\lambda_i-\lambda_j-r}t^{j-i}} \\
&= \prod_{i<j} \Big(\Big(\prod_{r=0}^{k-1} \frac{(1-q^{\lambda_i-\lambda_j+r}t^{j-i})}{1}\Big)\Big(\prod_{r=-k}^{1} \frac{1}{(1-q^{\lambda_i-\lambda_j+1+r}t^{j-i})}\Big)\Big) \\
&= \prod_{i<j} \frac{(q^{\lambda_i-\lambda_j}t^{j-i};q)_\infty (q^{\lambda_i-\lambda_j+1}t^{j-i};q)_\infty}{(q^{\lambda_i-\lambda_j}t^{j-i}q^k;q)_\infty (q^{\lambda_i-\lambda_j+1}t^{j-i}q^{-k};q)_\infty} \\
&= \prod_{i<j} \frac{(q^{\lambda_i-\lambda_j}t^{j-i};q)_\infty (qq^{-(-\lambda_i+\lambda_j)}t^{j-i};q)_\infty}{(tq^{\lambda_i-\lambda_j}t^{j-i};q)_\infty (t^{-1}qq^{-(-\lambda_i+\lambda_j)}t^{j-i};q)_\infty} ? =? \mathrm{ev}^t_\lambda(\Delta^+(t))\mathrm{ev}^{t^{-1}}_{-\lambda}(\Delta^-(t^{-1})).
\end{aligned}$$

Since this formula is true for $k \in \mathbb{Z}_{>0}$ then it is true for arbitrary q and t. (see [Mac, Ch. VI §9 Ex. 2(d)]) □

10.6 The symmetric inner product

Let $\mathbb{C}[X]=\mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$. Define involutions $^{-}\colon \mathbb{C}[X]\to\mathbb{C}[X]$, $^{\sigma}\colon \mathbb{C}[X]\to\mathbb{C}[X]$, $^{t}\colon \mathbb{C}[X]\to\mathbb{C}[X]$, by

$$\begin{array}{lll} ^{-}\colon \mathbb{C}[X]\to\mathbb{C}[X] & \text{by} & \bar{f}(x_1,\ldots,x_n;q,t)=f(x_1^{-1},\ldots,x_n^{-1};q^{-1},t^{-1}), \\ ^{\sigma}\colon \mathbb{C}[X]\to\mathbb{C}[X] & \text{by} & f^{\sigma}(x_1,\ldots,x_n;q,t)=f(x_1^{-1},\ldots,x_n^{-1};q,t), \\ ^{t}\colon \mathbb{C}[X]\to\mathbb{C}[X] & \text{by} & f^{t}(x_1,\ldots,x_n;q,t)=f(x_1,\ldots,x_n;q^{-1},t^{-1}). \end{array} \qquad \text{(invdefns)}$$

Define

$$\nabla_{q,t}=\prod_{i\neq j}\frac{(x_ix_j^{-1};q)_\infty}{(tx_ix_j^{-1};q)_\infty} \qquad\text{and}\qquad \Delta_{q,t}=\nabla_{q,t}\prod_{i<j}\frac{1-tx_i^{-1}x_j}{1-x_i^{-1}x_j}. \qquad \text{(DnabladefnGL)}$$

For $f\in\mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$ let

$$\mathrm{ct}(f)=(\text{constant term in } f).$$

Define two scalar products $(\ ,\ )\colon \mathbb{C}[X]\times\mathbb{C}[X]\to\mathbb{C}(q,t)$ and $\langle\ ,\ \rangle\colon \mathbb{C}[X]\times\mathbb{C}[X]\to\mathbb{C}(q,t)$ by

$$(f_1,f_2)_{q,t}=\mathrm{ct}(f_1\overline{f_2}\Delta_{q,t}) \qquad\text{and}\qquad \langle f_1,f_2\rangle_{q,t}=\frac{1}{|W_0|}\mathrm{ct}(f_1f_2^{\sigma}\nabla_{q,t}). \qquad \text{(innproddefnB)}$$

Proposition 10.10 provides a comparison of $(,)_{q,t}$ and $\langle,\rangle_{q,t}$ as inner products on symmetric functions.

Proposition 10.10. *Let $f,g\in\mathbb{C}[X]^{W_0}$. Then*

$$\langle f,g\rangle_{q,t}=\frac{1}{W_0(t)}(f,g^t)_{q,t}. \qquad (10.6)$$

Proof. Let $f,g\in\mathbb{C}[X]^{W_0}$. Then

$$\begin{aligned} \langle f,g\rangle_{q,t} &=\frac{1}{|W_0|}\mathrm{ct}(fg^{\sigma}\nabla_{q,t}) && \text{(by (innproddefnB))}\\ &=\frac{1}{W_0(t)|W_0|}\mathrm{ct}(f\bar{g}^t\nabla_{q,t}W_0(t)) && \text{(by (invdefns))}\\ &=\frac{1}{W_0(t)|W_0|}\mathrm{ct}\Big((f\bar{g}^t\nabla_{q,t})\big(\sum_{w\in W_0}w(c_{w_0}(x^{-1};t))\Big) && \text{(by (Poinbysymm))}\\ &=\frac{1}{W_0(t)|W_0|}\mathrm{ct}\Big(\sum_{w\in W_0}w\big(f\bar{g}^t\nabla_{q,t}c_{w_0}(x^{-1};t)\big)\Big) && (f,g,\nabla_{q,t}\in\mathbb{C}[X]^{W_0})\\ &=\frac{1}{W_0(t)|W_0|}\mathrm{ct}\Big(\sum_{w\in W_0}w\big(f\bar{g}^t\Delta_{q,t}\big)\Big) && \text{(by (DnabladefnGL)}\\ &=\frac{1}{W_0(t)}\mathrm{ct}(f\bar{g}^t\Delta_{q,t}) && \text{(by (cttosymct))}\\ &=\frac{1}{W_0(t)}(f,g^t)_{q,t}. && \text{(by (innproddefnB))} \end{aligned}$$

□

10.7 Notes and references

In [Mac, Ch. §9], f^σ is denoted $\bar{f}$ and the constant term is defined in [Mac, Ch. VI §9]. The involutions are defined in [Mac03, (5.1.15),(5.1.30),(5.1.35)]. The constant term is defined in [Mac03, (5.1.8)]. In [Mac, Ch. VI §9 (9.2) and Ex. 1(a)], $\nabla_{q,t}$ is denoted $\Delta = \Delta(x; q, t)$ and $\Delta_{q,t}$ is denoted $\Delta'(x; q, t)$.) In [Mac, Ch. VI §9], the inner product $(f, g)_{q,t}$ is denoted $\langle f, g\rangle'$; the inner product $\langle f, g\rangle_{q,t}$ is not explicitly used though it appears implicitly in [Mac, Ch. VI §9 Ex. 1]. The values for $(1, 1)_{q,q^k}$ and $\langle 1, 1\rangle_{q,q^k}$ are as given in [Mac, Ch. VI §9 Ex. 1(c)] and [Mac, Ch. VI §9 Ex. 1(a)].