14 Lecture 1: Exercises, Remarks and Examples

14.1 HW from in person Lecture 1

HW 1. Recall the W-action on $\mathbb{Z}^n$. Let $\mathrm{Stab}(0,0,\ldots,0) = \{w \in W \mid w(0,0,\ldots,0)\} = (0,0,\ldots,0)\}$. Show that

$$W_0 = \mathrm{Stab}(0,0,\ldots,0).$$

HW 2. Show that

$$\{v \in S_n \mid \ell(v) = 1\} = \{s_1,\ldots,s_{n-1}\}.$$

HW 3. Show that if $w \in W$ then $\ell(w)$ is finite.

HW 4. Show that $\ell(\pi) = 0$ and $\ell(s_i) = 1$ for $i \in \{0,1,\ldots,n-1\}$.

HW 5. Show that

$$\{w \in W \mid \ell(w) = 0\} = \{\pi^k \mid k \in \mathbb{Z}\}.$$

HW 6. Show that

$$\{w \in W \mid \ell(w) = 1\} = \{\pi^k s_i \mid k \in \mathbb{Z} \text{ and } i \in \{0,1,\ldots,n-1\}\}.$$

HW 7. Let $Z(W) = \{z \in W \mid \text{if } w \in W \text{ then } zw = wz\}$. Show that

$$Z(W) = \{\pi^{kn} \mid k \in \mathbb{Z}\}.$$

HW 8. Show that $\pi^n = t_{(1,1,\ldots,1)}$.

HW 9. Let $\varepsilon_i = (0,\ldots,0,1,0,\ldots,0)$ with 1 in the ith spot. Show that

$$t_{(1,0,\ldots,0)} = \pi s_{n-1}\cdots s_1 \qquad \text{and} \qquad t_{\varepsilon_i} = s_{i-1}\cdots s_2 s_1 \pi s_{n-1}\cdots s_i.$$

HW 10. Is it true that $\pi t_\mu = t_{\pi\mu}\pi$?

HW 11. Let W^{ad} be the subgroup of W generated by $\{s_0,s_1,\ldots,s_{n-1}\}$. Show that

$$W^{\mathrm{ad}} = \{w \in W \mid w(1) + \cdots + w(n) = \tfrac{1}{2}n(n+1)\}.$$

HW 12. Explain why it is sensible to define $s_i = s_{i+n}$ for $i \in \mathbb{Z}$. Then show that

$$s_i^2 = 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad \pi s_i \pi^{-1} = s_{i+1}, \qquad \text{and} \qquad s_i s_j = s_j s_i,$$

for $i,j \in \mathbb{Z}$ with $j + n\mathbb{Z} \notin \{(i-1)+n\mathbb{Z}, (i+1)+n\mathbb{Z}\}$.

HW 13. Show that if $\mu \in \mathbb{Z}^n$ and $v \in S_n$ then $vt_\mu = t_{v\mu}v$.

HW 14. Show that

$$W = \{t_\mu v \mid \mu \in \mathbb{Z}^n \text{ and } v \in S_n\}.$$

HW 15. Show that

$$W = \{\pi^k u \mid k \in \mathbb{Z} \text{ and } u \in W^{\mathrm{ad}}\}.$$

HW 16. For $\mu \in \mathbb{Z}^n$ let $v_\mu \in S_n$ be minimal length such that $v_\mu\mu$ is weakly decreasing. Show that $v_{(0,4,5,1,4)} = s_4 s_2 s_3$.

HW 17. For $\mu \in \mathbb{Z}^n$ let $v_\mu \in S_n$ be minimal length such that $v_\mu\mu$ is weakly decreasing. Show that if $r \in \{1,\ldots,n\}$ then

$$v_\mu(r) = \#\{r' > r \mid \mu_{r'} < \mu_r\} + \#\{r' < r \mid \mu_{r'} \le \mu_r\}.$$

HW 18. Define $u_\mu = t_\mu v_\mu^{-1}$. Show that $u_\mu W_0 = t_\mu W_0$ and

$$u_\mu \text{ is the unique minimal length element in the coset } t_\mu W_0.$$

14.2 Examples from Supplement

14.2.1 Examples of the inversion set $\mathrm{Inv}(w)$.

Define n-periodic permutations π and $s_0, s_1, \ldots, s_{n-1} \in W$ by

$$\pi(i) = i+1, \quad \text{for } i \in \mathbb{Z}, \tag{14.1}$$

$$\begin{matrix} s_i(i) = i+1, \\ s_i(i+1) = i, \end{matrix} \qquad \text{and} \quad s_i(j) = j \ \text{ for } j \in \{0,1,\ldots,i-1,i+2,\ldots,n-1\}. \tag{14.2}$$

An *inversion* of a bijection $w\colon \mathbb{Z} \to \mathbb{Z}$ is

$$(j,k) \in \mathbb{Z}\times\mathbb{Z} \quad \text{with} \quad j<k \text{ and } w(j) > w(k).$$

and the affine root corresponding to an inversion

$$(i,k) = (i, j+\ell n) \quad \text{with } i,j \in \{1,\ldots,n\} \text{ and } \ell \in \mathbb{Z}, \qquad \text{is} \quad \beta^\vee = \varepsilon_i^\vee - \varepsilon_j^\vee + \ell K. \tag{14.3}$$

Let $n=3$. The element

$$w = s_1s_2 \quad \text{has} \quad w(1)=2,\ w(2)=3,\ w(3)=1,$$

and $w(1) > w(3)$ and $w(2) > w(3)$ and

$$\mathrm{Inv}(w) = \{\alpha_2^\vee, s_2\alpha_1^\vee\} = \{\varepsilon_2^\vee - \varepsilon_3^\vee, \varepsilon_1^\vee - \varepsilon_3^\vee\}.$$

The element

$$w = s_2s_1 \quad \text{has} \quad w(1)=3,\ w(2)=1,\ w(3)=2,$$

and $w(1) > w(2)$ and $w(1) > w(3)$ and

$$\mathrm{Inv}(w) = \{\alpha_1^\vee, s_1\alpha_2^\vee\} = \{\varepsilon_1^\vee - \varepsilon_2^\vee, \varepsilon_1^\vee - \varepsilon_3^\vee\}.$$

14.2.2 Relations in the affine Weyl group W

The following relations are useful when working with n-periodic permutations.

Proposition 14.1. *Then*

$$s_0 = t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1}\cdots s_2s_1s_2\cdots s_{n-1}, \qquad t_{\varepsilon_1^\vee} = \pi s_{n-1}\cdots s_2s_1, \tag{14.4}$$

$$\textit{and} \qquad t_{\varepsilon_{i+1}^\vee} = s_i t_{\varepsilon_i^\vee} s_i, \qquad \pi s_i \pi^{-1} = s_{i+1}, \tag{14.5}$$

for $i \in \{1,\ldots,n-1\}$.

Proof. Proof of (14.9): If $i \notin \{1,n\}$

$$t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1}\cdots s_2s_1s_2\cdots s_{n-1}(i) t_{\varepsilon_1^\vee - \varepsilon_n^\vee}(i) = i = s_0(i).$$

If $i=1$ then

$$t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1}\cdots s_2s_1s_2\cdots s_{n-1}(1) = t_{\varepsilon_1^\vee - \varepsilon_n^\vee}(n) = n-n = 0 = s_0(1),$$

and, if $i=n$ then

$$t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1}\cdots s_2s_1s_2\cdots s_{n-1}(n) = t_{\varepsilon_1^\vee - \varepsilon_n^\vee}(1) = 1+n = s_0(n),$$

For $i \in \{2, \ldots, n\}$

$$\pi s_{n-1} \cdots s_1(i) = \pi(i-1) = i = t_{\varepsilon_1}(i), \qquad \text{and} \qquad \pi s_{n-1} \cdots s_1(1) = \pi(n) = n+1 = t_{\varepsilon_1}(1).$$

Proof of (14.10):

$$\begin{aligned} s_i t_{\varepsilon_i^\vee} s_i(i) &= s_i t_{\varepsilon_i^\vee}(i+1) = s_i(i+1) = i = t_{\varepsilon_{i+1}^\vee}(i), \\ s_i t_{\varepsilon_i^\vee} s_i(i+1) &= s_i t_{\varepsilon_i^\vee}(i) = s_i(i+n) = i+1+n, = t_{\varepsilon_{i+1}^\vee}(i+1), \\ s_i t_{\varepsilon_i^\vee} s_i(j) &= s_i t_{\varepsilon_i^\vee}(j) = s_i(j) = j = t_{\varepsilon_{i+1}^\vee}(j), \quad \text{if } j \in \{1, \ldots, n\} \text{ and } j \notin \{i, i+1\}. \end{aligned}$$

Finally,

$$\begin{aligned} \pi s_i \pi^{-1}(i) &= \pi s_i(i-1) = \pi(i) = i+1 = s_{i+1}(i), \quad \text{and} \\ \pi s_i \pi^{-1}(i+1) &= \pi s_i(i) = \pi(i+1) = i+2 = s_{i+1}(i+1). \end{aligned}$$

□

14.2.3 The elements u_μ, v_μ, z_μ and t_μ.

Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ and let u_μ be the minimal length n-periodic permutation such that

$$u_\mu(0, 0, \ldots, 0) = (\mu_1, \ldots, \mu_n).$$

Let $\lambda = (\lambda, \ldots, \lambda_n)$ be the weakly decreasing rearrangement of μ and let

$$\begin{aligned} &z_\mu \in S_n \quad \text{be minimal length such that} \quad z_\mu \lambda = \mu, \quad \text{and let} \\ &v_\mu \in S_n \quad \text{be minimal length such that } v_\mu \mu \text{ is weakly increasing.} \end{aligned}$$

Let $t_\mu \colon \mathbb{Z} \to \mathbb{Z}$ be the n-periodic permutation determined by

$$t_\mu(1) = 1 + n\mu_1, \quad t_\mu(2) = 2 + n\mu_2, \quad \ldots, \quad t_\mu(n) = n + n\mu_n. \tag{14.6}$$

14.2.4 Relating u_μ, v_μ, z_μ to $u_\lambda, v_\lambda, z_\lambda$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$. Let $S_\lambda = \{w \in S_n \mid w\lambda = \lambda\}$ be the stabilizer of λ in S_n. Let

w_0 be the longest element in S_n,
w_λ the longest length element in S_λ, and
w^λ the minimal length element in the coset $w_0 S_\lambda$,

so that

$$w_0 = w^\lambda w_\lambda \quad \text{and} \quad \binom{n}{2} = \ell(w_0) = \ell(w^\lambda) + \ell(w_\lambda).$$

Let $\mu \in \mathbb{Z}^n$ and let λ be the decreasing rearrangement of λ. Let $z_\mu \in S_n$ be minimal length such that $\mu = z_\mu \lambda$. Then $z_\lambda = 1$,

$$t_\mu = u_\mu v_\mu = (z_\mu u_\lambda) v_\mu \qquad \text{and} \quad t_\lambda = u_\lambda v_\lambda = u_\lambda (w^\lambda)^{-1}, \quad \text{with}$$

$$\ell(t_\mu) = \ell(u_\mu) + \ell(v_\mu) = \ell(z_\mu) + \ell(u_\lambda) + \ell(v_\mu) \quad \text{and} \quad \ell(t_\lambda) = \ell(u_\lambda) + \ell((w^\lambda)^{-1}).$$

Using that $z_\mu t_\lambda z_\mu^{-1} = t_{z_\mu \lambda} = t_\mu$ gives that the elements u_μ and v_μ are given in terms of z_μ, u_λ and w^λ by

$$u_\mu = z_\mu u_\lambda \quad \text{and} \quad v_\mu = v_\lambda z_\mu^{-1} = (w^\lambda)^{-1} z_\mu^{-1} = (z_\mu w^\lambda)^{-1} = (z_\mu w_0 w_\lambda)^{-1} = w_\lambda w_0 z_\mu^{-1},$$

since $v_\lambda = (w^\lambda)^{-1}$ and $v_\lambda = v_\mu z_\mu$ with $\ell((w_\lambda)^{-1}) = \ell(v_\lambda) = \ell(v_\mu) + \ell(z_\mu)$.

14.2.5 Inversions of t_{ε_1}, $t_{-\varepsilon_1}$ and t_{ε_2}

Let t_μ be as in (14.6) and let $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 appears in the ith position. Then

$$\begin{aligned}
t_{\varepsilon_1} &= (1_1, 0_2, \ldots, 0_n) = \begin{pmatrix} 1 & 2 & \cdots & n \\ n+1 & 2 & \cdots & n \end{pmatrix} = \pi s_{n-1} \cdots s_1, \\
t_{-\varepsilon_1} &= (-1_1, 0_2, \ldots, 0_n) = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1-n & 2 & \cdots & n \end{pmatrix} = s_1 \cdots s_{n-1} \pi^{-1}, \\
t_{\varepsilon_1} s_1 &= (0_2, 1_1, 0_3, \ldots, 0_n) = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 1+n & 3 & \cdots & n \end{pmatrix} = \pi s_{n-1} \cdots s_2, \\
s_1 t_{\varepsilon_1} &= (1_2, 0_1, 0_3, \ldots, 0_n) = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2+n & 1 & 3 & \cdots & n \end{pmatrix} = s_1 \pi s_{n-1} \cdots s_1, \\
t_{\varepsilon_2} = s_1 t_{\varepsilon_1} s_1 &= (0_1, 1_2, 0_3, \ldots, 0_n) = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2+n & 3 & \cdots & n \end{pmatrix} = s_1 \pi s_{n-1} \cdots s_2,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Inv}(t_{\varepsilon_1}) &= \{(1,2), (1,3), \ldots, (1,n)\} \\
&= \{\alpha_1^\vee, s_1\alpha_2^\vee, \ldots, s_1 \cdots s_{n-2}\alpha_{n-1}^\vee\} = \{\varepsilon_1^\vee - \varepsilon_2^\vee, \varepsilon_1^\vee - \varepsilon_3^\vee, \ldots, \varepsilon_1^\vee - \varepsilon_n^\vee\} \\
\mathrm{Inv}(t_{-\varepsilon_1}) &= \{(2-n, 1), (3-n, 1), \ldots, (n-n, 1)\} = \{(n, 1+n), (n-1, 1+n), \ldots, (2, 1+n)\} \\
&= \{\pi\alpha_{n-1}^\vee, \pi s_{n-1}\alpha_{n-2}^\vee, \ldots, \pi s_{n-1} \cdots s_2 \alpha_1^\vee\} \\
&= \{\varepsilon_n^\vee - (\varepsilon_1^\vee - K), \varepsilon_{n-1}^\vee - (\varepsilon_1^\vee - K), \ldots \varepsilon_2^\vee - (\varepsilon_1^\vee - K)\} \\
\mathrm{Inv}(t_{\varepsilon_1} s_1) &= \{(2,3), \ldots, (2,n)\} \\
&= \{\alpha_2^\vee, s_2\alpha_3^\vee, \ldots, s_2 \cdots s_{n-2}\alpha_{n-1}^\vee\} = \{\varepsilon_2^\vee - \varepsilon_3^\vee, \varepsilon_2^\vee - \varepsilon_4^\vee, \ldots, \varepsilon_2^\vee - \varepsilon_n^\vee\} \\
\mathrm{Inv}(s_1 t_{\varepsilon_1}) &= \{(1,2), (1,3), \ldots, (1,n), (1-n, 2)\} = \{(1,2), (1,3), \ldots, (1,n), (1, 2+n)\} \\
&= \{\alpha_1^\vee, s_1\alpha_2^\vee, \ldots, s_1 \cdots s_{n-2}\alpha_{n-1}^\vee, s_1 \cdots s_{n-2} s_{n-1} \pi^{-1} \alpha_1^\vee\} \\
&= \{\varepsilon_1^\vee - \varepsilon_2^\vee, \varepsilon_1^\vee - \varepsilon_3^\vee, \ldots, \varepsilon_1^\vee - \varepsilon_n^\vee, (\varepsilon_1^\vee + K) - \varepsilon_2^\vee\} \\
\mathrm{Inv}(t_{\varepsilon_2}) &= \{((2,3), \ldots, (2,n), (2-n, 1)\} = \{((2,3), \ldots, (2,n), (2, 1+n)\} \\
&= \{\alpha_2^\vee, s_2\alpha_3^\vee, \ldots, s_2 \cdots s_{n-2}\alpha_{n-1}^\vee, s_2 \cdots s_{n-2} s_{n-1} \pi^{-1} \alpha_1^\vee\} \\
&= \{\varepsilon_2^\vee - \varepsilon_3^\vee, \varepsilon_2^\vee - \varepsilon_4^\vee, \ldots, \varepsilon_2^\vee - \varepsilon_n^\vee, (\varepsilon_2^\vee + K) - \varepsilon_1^\vee\},
\end{aligned}$$

where we have used

$$s_1 \cdots s_{n-1} \pi^{-1} \alpha_1^\vee = s_1 \cdots s_{n-1} \pi^{-1} (\varepsilon_1^\vee - \varepsilon_2^\vee) = s_1 \cdots s_{n-1}((\varepsilon_n^\vee + K) - \varepsilon_1^\vee) = (\varepsilon_1^\vee + K) - \varepsilon_2^\vee, \quad \text{and}$$
$$s_2 \cdots s_{n-1} \pi^{-1} \alpha_1^\vee = s_2 \cdots s_{n-1}((\varepsilon_n^\vee + K) - \varepsilon_1^\vee) = (\varepsilon_2^\vee + K) - \varepsilon_1^\vee.$$

14.2.6 The elements u_μ and v_μ for $\mu = (0, 4, 5, 1, 4)$

Let u_μ, v_μ, z_μ and t_μ be as in Section 14.2.3. If $\mu = (0, 4, 5, 1, 4)$ then $\lambda = (5, 4, 4, 1, 0)$, and

$$z_\mu = s_2 s_4 s_1 s_2 s_3 s_4 \quad \text{since} \quad (5,4,4,1,0) \overset{s_1 s_2 s_3 s_4}{\to} (0,5,4,4,1) \overset{s_4}{\to} (0,5,4,1,4) \overset{s_2}{\to} (0,4,5,1,4),$$

$$v_\mu = s_4 s_2 s_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix} \qquad \text{with} \qquad \begin{aligned} v_\mu(1) &= 1 = 1, \\ v_\mu(2) &= 3 = 1 + \#\{1\}, \\ v_\mu(3) &= 5 = 1 + \#\{1,2\} + \#\{4\}, \\ v_\mu(4) &= 2 = 1 + \#\{1\}, \\ v_\mu(5) &= 4 = 1 + \#\{2,4\}, \end{aligned}$$

Then $v_\mu = (0_1, 0_3, 0_5, 0_3, 0_4)$ and

$$\mathrm{Inv}(v_\mu) = \{(2,4),(3,4),(3,5)\} = \{\alpha_3^\vee, s_3\alpha_2^\vee, s_3s_2\alpha_4^\vee\} = \{\varepsilon_3^\vee - \varepsilon_4^\vee, \varepsilon_2^\vee - \varepsilon_4^\vee, \varepsilon_3^\vee - \varepsilon_5^\vee\}.$$

Then, with $n = 5$,

$$v_\mu^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix} = (0_1, 0_4, 0_2, 0_5, 0_3) \quad \text{and}$$

$$u_\mu = t_\mu v_\mu^{-1} = (0_1, 4_3, 5_5, 1_2, 4_4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4+n & 2+4n & 5+4n & 3+5n \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 10 & 25 & 28 \end{pmatrix}$$

Then

$$\ell(t_\lambda) = \begin{pmatrix} \quad (5-4)+(5-4)+(5-1)+(5-0) \\ +(4-4)+(4-1)+(4-0) \\ +(4-1)+(4-0) \\ +(1-0) \end{pmatrix} = 26 = \ell(t_\mu) = \ell(u_\mu) + \ell(v_\mu)$$

with

$$\ell(u_\mu) = 6 + 7\cdot 2 + 3 = 23, \quad \ell(v_\mu) = 3, \quad \ell(z_\mu) = 6.$$

The decreasing rearrangement of $\mu = (0,4,5,1,4)$ is $\lambda = (5,4,4,1,0)$ and

$$z_\lambda = 1, \quad w_\lambda = s_2, \quad v_\lambda = w_0 s_2$$

14.2.7 The box greedy reduced word for u_μ.

If $\mu = (0,4,5,1,4)$ then the box greedy reduced word for u_μ is

$$u_\mu^\square = (s_1\pi)^6 (s_2s_1\pi)^7 (s_3s_2s_1\pi) = \begin{array}{|lllll} \boxed{s_1\pi} & \boxed{s_1\pi} & \boxed{s_2s_1\pi} & \boxed{s_2s_1\pi} & \\ \boxed{s_1\pi} & \boxed{s_1\pi} & \boxed{s_2s_1\pi} & \boxed{s_2s_1\pi} & \boxed{s_3s_2s_1\pi} \\ \boxed{s_1\pi} & & & & \\ \boxed{s_1\pi} & \boxed{s_2s_1\pi} & \boxed{s_2s_1\pi} & \boxed{s_2s_1\pi} & \end{array} \tag{14.7}$$

and the length of u_μ is

$$\ell(u_\mu) = 6 + 14 + 3 = 23, \qquad \text{since} \quad \ell(\pi) = 0 \quad \text{and} \quad \ell(s_i) = 1.$$

Using one-line notation for n-periodic permutations, the computation verifying the expression for $u_\mu^\square$ is

$$(0_1,4_3,5_5,1_2,4_4) \xrightarrow{s_1} (4_3,0_1,5_5,1_2,4_4) \xrightarrow{\pi^{-1}}$$
$$(0_1,5_5,1_2,4_4,3_3)) \xrightarrow{s_1} (5_5,0_1,1_2,4_4,3_3)) \xrightarrow{\pi^{-1}}$$
$$(0_1,1_2,4_4,3_3,4_5)) \xrightarrow{s_1} (1_2,0_1,4_4,3_3,4_5)) \xrightarrow{\pi^{-1}}$$
$$(0_1,4_4,3_3,4_5,0_2)) \xrightarrow{s_1} (4_4,0_1,3_3,4_5,0_2)) \xrightarrow{\pi^{-1}}$$
$$(0_1,3_3,4_5,0_2,3_4)) \xrightarrow{s_1} (3_3,0_1,4_5,0_2,3_4)) \xrightarrow{\pi^{-1}}$$
$$(0_1,4_5,0_2,3_4,2_3)) \xrightarrow{s_1} (4_5,0_1,0_2,3_4,2_3)) \xrightarrow{\pi^{-1}}$$
$$(0_1,0_2,3_4,2_3,3_5)) \xrightarrow{s_2} (0_1,3_4,0_2,2_3,3_5)) \xrightarrow{s_1} (3_4,0_1,0_2,2_3,3_5)) \xrightarrow{\pi^{-1}}$$
$$(0_1,0_2,2_3,3_5,2_4)) \xrightarrow{s_2} (0_1,2_3,0_2,3_5,2_4)) \xrightarrow{s_1} (2_3,0_1,0_2,3_5,2_4)) \xrightarrow{\pi^{-1}}$$
$$(0_1,0_2,3_5,2_4,1_3)) \xrightarrow{s_2} (0_1,3_5,0_2,2_4,1_3)) \xrightarrow{s_1} (3_5,0_1,0_2,2_4,1_3)) \xrightarrow{\pi^{-1}}$$
$$(0_1,0_2,2_4,1_3,2_5)) \xrightarrow{s_2} (0_1,2_4,0_2,1_3,2_5)) \xrightarrow{s_1} (2_4,0_1,0_2,1_3,2_5)) \xrightarrow{\pi^{-1}}$$
$$(0_1,0_2,1_3,2_5,1_4)) \xrightarrow{s_2} (0_1,1_3,0_2,2_5,1_4)) \xrightarrow{s_1} (1_3,0_1,0_2,2_5,1_4)) \xrightarrow{\pi^{-1}}$$
$$(0_1,0_2,2_5,1_4,0_3)) \xrightarrow{s_2} (0_1,2_5,0_2,1_4,0_3)) \xrightarrow{s_1} (2_5,0_1,0_2,1_4,0_3)) \xrightarrow{\pi^{-1}}$$
$$(0_1,0_2,1_4,0_3,1_5)) \xrightarrow{s_2} (0_1,1_4,0_2,0_3,1_5)) \xrightarrow{s_1} (1_4,0_1,0_2,0_3,1_5)) \xrightarrow{\pi^{-1}}$$
$$(0_1,0_2,0_3,1_5,0_4)) \xrightarrow{s_3} (0_1,0_2,1_5,0_3,0_4)) \xrightarrow{s_2} (0_1,1_5,0_2,0_3,0_4)) \xrightarrow{s_1} (1_5,0_1,0_2,0_3,0_4)) \xrightarrow{\pi^{-1}} (0_1,0_2,0_3,0_4,0_5))$$

14.2.8 Inversions of u_μ.

If $\mu = (0,4,5,1,4)$ then the inversion set of u_μ is

$\mathrm{Inv}(u_\mu) =$

<table>
<tr><td>$\varepsilon_3^\vee - \varepsilon_1^\vee + 4K$</td><td>$\varepsilon_3^\vee - \varepsilon_1^\vee + 3K$</td><td>$\varepsilon_3^\vee - \varepsilon_1^\vee + 2K$
$\varepsilon_3^\vee - \varepsilon_2^\vee + 2K$</td><td>$\varepsilon_3^\vee - \varepsilon_1^\vee + K$
$\varepsilon_3^\vee - \varepsilon_2^\vee + K$</td><td></td></tr>
<tr><td>$\varepsilon_5^\vee - \varepsilon_1^\vee + 5K$</td><td>$\varepsilon_5^\vee - \varepsilon_1^\vee + 4K$</td><td>$\varepsilon_5^\vee - \varepsilon_1^\vee + 3K$
$\varepsilon_5^\vee - \varepsilon_2^\vee + 3K$</td><td>$\varepsilon_5^\vee - \varepsilon_1^\vee + 2K$
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$\varepsilon_4^\vee - \varepsilon_2^\vee + 2K$</td><td>$\varepsilon_4^\vee - \varepsilon_1^\vee + K$
$\varepsilon_4^\vee - \varepsilon_2^\vee + K$</td><td></td></tr>
</table>

The following is an example that executes the last line of the proof of [GR21, Proposition 2.2]. The factor of s_1 in the factorization $u_\mu = s_1\pi u_{(0,5,1,4,3)}$ gives the root

$$\begin{aligned} u^{-1}_{(0,5,1,4,3)}\pi^{-1}(\varepsilon_1^\vee - \varepsilon_2^\vee) &= u^{-1}_{(0,5,1,4,3)}\pi^{-1}(\varepsilon_1^\vee - \varepsilon_2^\vee) = u^{-1}_{(0,5,1,4,3)}((\varepsilon_5^\vee + K) - \varepsilon_1^\vee) \\ &= v_{(0,5,1,4,3)}t^{-1}_{(0,5,1,4,3)}(\varepsilon_5^\vee - \varepsilon_1^\vee + K) = v_{(0,5,1,4,3)}(\varepsilon_5^\vee + 3K - (\varepsilon_1^\vee + 0K) + K) \\ &= \varepsilon_3^\vee - \varepsilon_1^\vee + 4K, \qquad \text{since } v_{(0,5,1,4,3)}(5) = 3. \end{aligned}$$

14.2.9 The column-greedy reduced word for u_μ.

Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$. Let $J = (j_1 < \ldots < j_r)$ be the sequence of positions of the nonzero entries of μ and let ν be the composition defined by

$$\nu_j = \mu_j - 1 \quad \text{if } j \in J \qquad \text{and} \qquad \nu_k = 0 \quad \text{if } k \notin J,$$

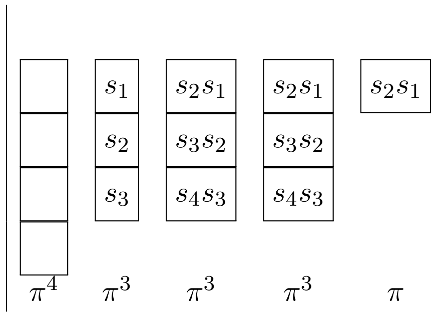
so that ν is the composition which has one fewer box than μ in each (nonempty) row. Define the *column-greedy reduced word* for the element u_μ inductively by setting

$$u_\mu^\downarrow = \Big(\prod_{m=1}^{r} s_{j_m-1}\cdots s_{m+1}s_m\Big)\pi^r u_\nu^\downarrow, \tag{14.8}$$

where the product is taken in increasing order.

For example, if $\lambda = (5,4,4,1,0)$ then $z_\lambda = 1$, $w_\lambda = s_2$, $v_\lambda = w_0 s_2$ and the column greedy reduced word for u_λ is

$$u_\lambda^\downarrow = \pi^4 s_1s_2s_3\pi^3(s_2s_1s_3s_2s_4s_3\pi^3)^2 s_2s_1\pi =$$



The computation verifying the expression for $u_\lambda^\downarrow$ is

$$\begin{array}{rl} & (5,4,4,1,0) \stackrel{\pi^{-4}}{\to} \\ (0,4,3,3,0) \stackrel{s_1s_2s_3}{\to} & (4,3,3,0,0) \stackrel{\pi^{-3}}{\to} \\ (0,0,3,2,2) \stackrel{s_2s_1s_3s_2s_4s_3}{\to} & (3,2,2,0,0) \stackrel{\pi^{-3}}{\to} \\ (0,0,2,1,1) \stackrel{s_2s_1s_3s_2s_4s_3}{\to} & (2,1,1,0,0) \stackrel{\pi^{-3}}{\to} \\ (0,0,2,0,0) \stackrel{s_2s_1}{\to} & (1,0,0,0,0) \stackrel{\pi^{-1}}{\to} (0,0,0,0,0) \end{array}$$

If $\mu = (0,4,5,1,4)$ then the column greedy reduced word for u_μ is

$$u_\mu^\downarrow = s_1s_2s_3s_4\pi^4 \cdot s_1s_2s_4s_3\pi^3 \cdot s_2s_1s_3s_2s_4s_3\pi^3 \cdot s_2s_1s_3s_2s_4s_3\pi^3 \cdot s_3s_2s_1\pi.$$

This follows from (14.7) by using that $\pi s_i \pi^{-1} = s_{i+1}$.

14.3 Presentations

Proposition 14.2. *Then*

$$s_0 = t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}, \qquad t_{\varepsilon_1^\vee} = \pi s_{n-1} \cdots s_2 s_1, \tag{14.9}$$

$$\text{and} \qquad t_{\varepsilon_{i+1}^\vee} = s_i t_{\varepsilon_i^\vee} s_i, \qquad \pi s_i \pi^{-1} = s_{i+1}, \tag{14.10}$$

for $i \in \{1, \ldots, n-1\}$.

Proof. Proof of (14.9): If $i \notin \{1, n\}$

$$t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}(i) t_{\varepsilon_1^\vee - \varepsilon_n^\vee}(i) = i = s_0(i).$$

If $i = 1$ then

$$t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}(1) = t_{\varepsilon_1^\vee - \varepsilon_n^\vee}(n) = n - n = 0 = s_0(1),$$

and, if $i = n$ then

$$t_{\varepsilon_1^\vee - \varepsilon_n^\vee} s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1}(n) = t_{\varepsilon_1^\vee - \varepsilon_n^\vee}(1) = 1 + n = s_0(n),$$

For $i \in \{2, \ldots, n\}$

$$\pi s_{n-1} \cdots s_1(i) = \pi(i-1) = i = t_{\varepsilon_1}(i), \qquad \text{and}$$
$$\pi s_{n-1} \cdots s_1(1) = \pi(n) = n + 1 = t_{\varepsilon_1}(1).$$

Proof of (14.10):

$$s_i t_{\varepsilon_i^\vee} s_i(i) = s_i t_{\varepsilon_i^\vee}(i+1) = s_i(i+1) = i = t_{\varepsilon_{i+1}^\vee}(i),$$
$$s_i t_{\varepsilon_i^\vee} s_i(i+1) = s_i t_{\varepsilon_i^\vee}(i) = s_i(i+n) = i + 1 + n, = t_{\varepsilon_{i+1}^\vee}(i+1),$$
$$s_i t_{\varepsilon_i^\vee} s_i(j) = s_i t_{\varepsilon_i^\vee}(j) = s_i(j) = j = t_{\varepsilon_{i+1}^\vee}(j), \quad \text{if } j \in \{1, \ldots, n\} \text{ and } j \notin \{i, i+1\}.$$

Finally,

$$\pi s_i \pi^{-1}(i) = \pi s_i(i-1) = \pi(i) = i + 1 = s_{i+1}(i), \quad \text{and}$$
$$\pi s_i \pi^{-1}(i+1) = \pi s_i(i) = \pi(i+1) = i + 2 = s_{i+1}(i+1).$$

□

14.4 The "affine Weyl group" and the "extended affine Weyl group"

The type GL_n affine Weyl group W is generated by $s_1, \ldots, s_n$ and π. The group W contains also s_0 and all the elements t_μ for $\mu \in \mathbb{Z}^n$. The *projection homomorphism* is the group homomorphism $\overline{}: W \to S_n$ given by

$$\overline{t_\mu v} = v, \qquad \text{for } \mu \in \mathbb{Z}^n \text{ and } v \in S_n. \tag{14.11}$$

The *type* PGL_n*-affine Weyl group* is the subgroup W_{PGL_n} generated by $s_0, s_1, \ldots, s_{n-1}$.

$$W_{PGL_n} = \{t_\mu v \mid \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n \text{ with } \mu_1 + \cdots + \mu_n = 0 \text{ and } v \in S_n\}, \qquad \text{and}$$
$$W_{GL_n} = W = \{t_\mu v \mid \mu \in \mathbb{Z}^n, v \in S_n\} = \{\pi^h w \mid h \in \mathbb{Z}, w \in W_{PGL_n}\}.$$

Then

$$W_{GL_n} = \mathbb{Z}^n \rtimes S_n = \Omega \ltimes W_{PGL_n}, \qquad \text{where} \quad \Omega = \{\pi^h \mid h \in \mathbb{Z}\} \quad \text{with} \quad \Omega \cong \mathbb{Z}.$$

The symbols $\ltimes$ and $\rtimes$ are brief notations whose purpose is to indicate that the relations in (14.10) hold.

The group W_{PGL_n} is also a quotient of W_{GL_n}, by the relation $\pi = 1$. The *type SL_n affine Weyl group* is the quotient of W_{GL_n} by the relation $\pi^n = 1$. This is equivalent to putting a relation requiring

$$t_\mu = t_\nu \qquad \text{if } \mu_i = \nu_i \bmod n \text{ for } i \in \{1, \ldots, n\}.$$

As explained in [St67, Ch. 3, Exercise after Corollary 5], there is a Chevalley group G_d for each positive integer d dividing n. The group G_d is a central extension of PGL_n by $\mathbb{Z}/d\mathbb{Z}$ (so that $G_1 = PGL_n$ and $G_n = SL_n$). Each of these groups G_d has an affine Weyl group W_{G_d}. The group W_{G_d} is the quotient of W_{GL_n} by the relation $\pi^d = 1$, and is an extension of W_{PGL_n} by $\mathbb{Z}/d\mathbb{Z}$. The group W_{PGL_n} is sometimes called the "affine Weyl group of type A" and the groups W_{GL_n} and W_{G_d} for $d \neq 1$ are sometimes called the "extended affine Weyl groups of type A". We prefer the more specific terminologies "affine Weyl group of type PGL_n" for W_{PGL_n}, "affine Weyl group of type SL_n" for W_{SL_n}, "affine Weyl group of type GL_n" for W_{GL_n}, and "affine Weyl group of type $PGL_n \veebar (\mathbb{Z}/d\mathbb{Z})$" for W_{G_d} (the symbol $\veebar$ indicates a central extension).