22 Lecture 4: Proofs

22.1 Lecture 4: Proof of the symmetrizer formula

Proposition 22.1. *As operators on* $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{-1}]$,

$$\mathbf{1}_0 = p_0 c_{w_0}(x^{-1}).$$

Proof. Let $w \in S_n$. Using $T_i = s_i c_{i,i+1}(x^{-1}) + (t^{\frac{1}{2}} - c_{i,i+1}(x))$ and a reduced word $w = s_{i_1} \cdots s_{i_\ell}$ and expanding, gives

$$T_w = T_{i_1} \cdots T_{i_\ell} = T_w = w c_w(x^{-1}) + \sum_{v<w} v b_v(x), \qquad \text{with } b_v(x) \in \mathbb{C}(x_1, \ldots, x_n).$$

Thus there are $a_v(x) \in \mathbb{C}(x_1, \ldots, x_n)$ such that

$$\mathbf{1}_0 = \sum_{w \in S_n} t^{-\frac{1}{2}\ell(w_0 w)} T_w = w_0 c_{w_0}(x^{-1}) + \sum_{w<w_0} v a_v(x), \qquad \text{(topterm)}$$

Since $p_0 = \sum_{w \in S_n} w$ then $s_i p_0 = p_0$ and

$$\begin{aligned} T_i(p_0 c_{w_0}(x^{-1})) &= \big(c_{i,i+1}(x) s_i + (t^{\frac{1}{2}} - c_{i,i+1}(x))\big) p_0 c_{w_0}(x^{-1}) \\ &= \big(c_{i,i+1}(x) + (t^{\frac{1}{2}} - c_{i,i+1}(x))\big) p_0 c_{w_0}(x^{-1}) = t^{\frac{1}{2}} \big(p_0 c_{w_0}(x^{-1})\big). \end{aligned}$$

Since $\mathbf{1}_0$ is determined, up to multiplication by a constant, by the property that $T_i \mathbf{1}_0 = t^{\frac{1}{2}} \mathbf{1}_0$ for $i \in \{1, \ldots, n-1\}$, it follows from (topterm) that, as operators on $\mathbb{C}[X]$,

$$\mathbf{1}_0 = p_0 c_{w_0}(x^{-1}).$$

□

22.2 Lecture 4: Proof of the general symmetrizer formula

Proposition 22.2. *Let* $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{Z}^n$ *and let* $v_\lambda \in S_n$ *be minimal length such that* $v_\lambda \lambda$ *is weakly increasing. Use notations for the symmetrizers and c-functions as in* (fullsymm), (partialWsymm), (partialHsymm) *and* (cfcnxw). *Then, as operators on* $\mathbb{C}[X]$,

$$\mathbf{1}_0 = p^\lambda c_{v_\lambda}(x^{-1}) \mathbf{1}_\lambda.$$

Proof. For $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$,

$$\mathrm{Inv}(w_\lambda) = \{(i,j) \mid i<j \text{ and } \lambda_i = \lambda_j \} \qquad \text{and} \qquad \mathrm{Inv}(v_\lambda) = \{(i,j) \mid i<j \text{ and } \lambda_i > \lambda_j \}.$$

If $u \in W_\lambda$ then $\lambda_{u(i)} > \lambda_{u(j)}$ if $\lambda_i > \lambda_j$ so that $u\mathrm{Inv}(v_\lambda) = \{(u(i), u(j)) \mid i<j \text{ and } \lambda_i > \lambda_j\} = \mathrm{Inv}(v_\lambda)$, which gives that $w_\lambda^{-1} c_{v_\lambda} = u c_{v_\lambda} = c_{v_\lambda}$ for $u \in W_\lambda$. This is the reason for the equalities

$$c_{w_0} = c_{v_\lambda w_\lambda} = (w_\lambda^{-1} c_{v_\lambda}) c_{w_\lambda} = c_{v_\lambda} c_{w_\lambda} \qquad \text{and} \qquad p_\lambda c_{v_\lambda} = c_{v_\lambda} p_\lambda. \qquad \text{(cfcnsplit)}$$

Replacing S_n by the group W_λ in the proof of Proposition 4.1 gives $\mathbf{1}_\lambda = p_\lambda c_{w_\lambda}(x^{-1})$. Using the relations in (cfcnsplit) and the identity $\mathbf{1}_\lambda = p_\lambda c_{w_\lambda}(x^{-1})$ gives

$$\mathbf{1}_0 = p_0 c_{w_0}(x^{-1}) = p^\lambda p_\lambda (w_\lambda^{-1} c_{v_\lambda}(x^{-1})) c_{w_\lambda}(x^{-1}) = p^\lambda c_{v_\lambda}(x^{-1}) p_\lambda c_{w_\lambda}(x^{-1}) = p^\lambda c_{v_\lambda}(x^{-1}) \mathbf{1}_\lambda.$$

□

22.3 Lecture 4: Proof of the E-expansion of P_λ

Proposition 22.3. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ and let $S_n\lambda$ be the set of distinct rearrangements of λ. Then*

$$P_\lambda = \sum_{\mu \in S_n} t^{\#\{i<j \ | \ \mu_i > \mu_j\}} \Big(\prod_{\substack{1\le i<j\le n \\ \mu_i>\mu_j}} \frac{1-q^{\mu_i-\mu_j}t^{j-i-1}}{1-q^{\mu_i-\mu_j}t^{j-i}} \Big) E_\mu$$

Proof. The symmetric Macdonald polynomial P_λ is the unique element of $\mathbb{C}[X]^\lambda$ such that $t^{\frac12}T_{s_i} = tP_\lambda$ for $i \in \{1, \ldots, n-1\}$. Write $P_\lambda = \sum_{\mu \in S_n\lambda} b_\mu E_\mu$. Then $t^{\frac12}T_{s_i}P_\lambda = tP_\lambda$ gives

$$\begin{aligned} \sum_{\mu\in S_n\lambda} tb_\mu E_\mu = tP_\lambda = t^{\frac12}T_{s_i} \sum_{\mu\in S_n\lambda} b_\mu E_\mu \\ = \sum_{\substack{\mu\in S_n\lambda \\ \mu_i>\mu_{i+1}}} b_\mu\Big(E_{s_i\mu} - \frac{1-t}{1-a_\mu}E_\mu\Big) + b_{s_i\mu}\Big(D_\mu E_\mu + \frac{1-t}{1-a_{s_i\mu}}E_{s_i\mu}\Big) + \sum_{\mu_i=\mu_{i+1}} tb_\mu E_\mu, \end{aligned}$$

and comparing coefficients of E_μ on each side of this equation gives that, if $\mu_i > \mu_{i+1}$ then

$$tb_{s_i\mu} = b_\mu + \frac{1-t}{1-a_{s_i\mu}} b_{s_i\mu} \qquad \text{so that} \qquad b_\mu = b_{s_i\mu}\Big(t - \frac{1-t}{1-a_{s_i\mu}}\Big).$$

Thus

$$b_\mu = b_{s_i\mu}\Big(\frac{1-q^{\mu_{i+1}-\mu_i}t^{v_\mu(i+1)-v_\mu(i)+1}}{1-q^{\mu_{i+1}-\mu_i}t^{v_\mu(i+1)-v_\mu(i)}}\Big) = b_{s_i\mu}\Big(t\frac{1-q^{\mu_i-\mu_{i+1}}t^{v_\mu(i)-v_\mu(i+1)-1}}{1-q^{\mu_i-\mu_{i+1}}t^{v_\mu(i)-v_\mu(i+1)}}\Big).$$

The result then follows, by recursion, from the base case $b_\gamma = 1$ where γ is the weakly increasing rearrangement of λ. □

22.4 Lecture 4: Second proof of the E-expansions

Proposition 22.4. *(E-expansions) Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \ge \cdots \ge \lambda_n$. Then*

$$\begin{aligned} P_\lambda &= \sum_{z\in W^\lambda} t^{\frac12\ell(v_\lambda z)} \mathrm{ev}^t_{z\lambda}(c_{v_\lambda z}(Y))E_{z\lambda} \qquad \textit{and} \\ A_{\lambda+\rho} &= \sum_{z\in W_0} (-t^{\frac12})^{\ell(w_0 z)} \mathrm{ev}^t_{z(\lambda+\rho)}(c_{w_0 z}(Y^{-1}))E_{z(\lambda+\rho)}. \end{aligned}$$

Alternatively,

$$\begin{aligned} P_\lambda &= \sum_{\mu\in W_0\lambda} \Big(\prod_{\substack{1\le i<j\le n \\ \mu_i>\mu_j}} t\Big(\frac{1-q^{\mu_i-\mu_j}t^{v_\mu(i)-v_\mu(j)-1}}{1-q^{\mu_i-\mu_j}t^{v_\mu(i)-v_\mu(j)}}\Big)\Big)E_\mu \qquad \textit{and} \\ A_{\lambda+\rho} &= \sum_{\mu\in W_0(\lambda+\rho)} \Big(\prod_{\substack{\alpha\in R^+ \\ \langle\mu,\alpha^\vee\rangle>0}} (-1)\Big(\frac{1-q^{\mu_i-\mu_j}t^{v_\mu(i)-v_\mu(j)+1}}{1-q^{\mu_i-\mu_j}t^{v_\mu(i)-v_\mu(j)}}\Big)\Big)E_\mu. \end{aligned}$$

Proof. Note that the coefficient of $E_{w_0\lambda}$ in P_λ is 1 and the coefficient of $E_{w_0(\lambda+\rho)}$ in $A_{\lambda+\rho}$ is 1.

For the first statement, (slicksymmA), (creationPA) and (PoinbysymmB) give

$$\begin{aligned}P_\lambda &= \frac{t^{\frac12\ell(w_0)}}{W_\lambda(t)}\mathbf{1}_0E_\lambda = \frac{t^{\frac12(\ell(w_0)-\ell(w_\lambda))}}{t^{-\frac12\ell(w_\lambda)}W_\lambda(t)}\Big(\sum_{z\in W^\lambda}\eta_z\Big)c_{v_\lambda}(Y)\mathbf{1}_\lambda E_\lambda = t^{\frac12(\ell(w_0)-\ell(w_\lambda))}\sum_{z\in W^\lambda}\tau_z^\vee\frac{c_{v_\lambda}(Y)}{c_z(Y)}E_\lambda \\ &= t^{\frac12\ell(v_\lambda)}\sum_{z\in W^\lambda}\tau_z^\vee(z^{-1}c_{v_\lambda z}(Y))E_\lambda = t^{\frac12\ell(v_\lambda)}\sum_{z\in W^\lambda}c_{v_\lambda z}(Y)\tau_z^\vee E_\lambda \\ &= t^{\frac12\ell(v_\lambda)}\sum_{z\in W^\lambda}\mathrm{ev}^t_{z\lambda}(c_{v_\lambda z}(Y)))t^{-\frac12\ell(z)}E_{z\lambda} = \sum_{z\in W^\lambda}t^{\frac12\ell(v_\lambda z)}\mathrm{ev}^t_{z\lambda}(c_{v_\lambda z}(Y)))E_{z\lambda}.\end{aligned}$$

If $z\in W^\lambda$ and $\mu=z\lambda$ then $\mathrm{Inv}(v_\lambda z)=\{(i,j)\mid 1\le i<j\le n \text{ and } \mu_i>\mu_j\}$ so that

$$\begin{aligned}\mathrm{ev}^t_\mu(c_{v_\lambda z}(Y)) &= \mathrm{ev}^t_\mu\Big(\prod_{\substack{1\le i<j\le n\\ \mu_i>\mu_j}}\frac{t^{-\frac12}-t^{\frac12}Y_iY_j^{-1}}{1-Y_iY_j^{-1}}\Big) = \mathrm{ev}^t_\mu\Big(\prod_{\substack{1\le i<j\le n\\ \mu_i>\mu_j}}t^{\frac12}\frac{t^{-1}Y_i^{-1}Y_j-1}{Y_i^{-1}Y_j-1}\Big) \\ &= t^{\frac12\ell(v_\lambda z)}\prod_{\substack{1\le i<j\le n\\ \mu_i>\mu_j}}\frac{1-q^{\mu_i-\mu_j}t^{v_\mu(i)-v_\mu(j)-1}}{1-q^{\mu_i-\mu_j}t^{v_\mu(i)-v_\mu(j)}}\end{aligned}$$

where we have used that $\mathrm{ev}^t_\mu(Y_i)=q^{-\mu_i}t^{-(v_\mu(i)-1)+\frac12(n-1)}$ so that

$$\mathrm{ev}^t_\mu(Y_i^{-1}Y_j) = q^{\mu_i}t^{(v_\mu(i)-1)-\frac12(n-1)}q^{-\mu_j}t^{-(v_\mu(j)-1)+\frac12(n-1)} = q^{\mu_i-\mu_j}t^{v_\mu(i)-v_\mu(j)}.$$

For the second statement,

$$\begin{aligned}A_{\lambda+\rho} &= t^{\frac12\ell(w_0)}\varepsilon_0E_{\lambda+\rho} = t^{\frac12\ell(w_0)}c_{w_0}(Y^{-1})\sum_{z\in W_0}\det(w_0z)\eta_zE_{\lambda+\rho} \\ &= t^{\frac12\ell(w_0)}\sum_{z\in W_0}\det(w_0z)c_{w_0z}(Y^{-1})t^{-\frac12\ell(z)}t^{\frac12\ell(z)}\tau_z^\vee E_{\lambda+\rho} \\ &= \sum_{z\in W_0}\det(w_0z)c_{w_0z}(Y^{-1})t^{\frac12\ell(w_0z)}E_{z(\lambda+\rho)} \\ &= \sum_{z\in W_0}(-1)^{\ell(w_0z)}t^{\frac12\ell(w_0z)}\mathrm{ev}^t_{z(\lambda+\rho)}(c_{w_0z}(Y^{-1}))E_{z(\lambda+\rho)}.\end{aligned}$$

If $\mu=z(\lambda+\rho)$ then

$$\mathrm{ev}^t_\mu(c_{w_0z}(Y^{-1})) = \prod_{\substack{1\le i<j\le n\\ \mu_i>\mu_j}}\frac{t^{-\frac12}-t^{\frac12}Y_i^{-1}Y_j}{1-Y_i^{-1}Y_j} = t^{-\frac12\ell(w_0z)}\prod_{\substack{1\le i<j\le n\\ \mu_i>\mu_j}}\frac{1-tq^{\mu_i-\mu_j}t^{v_\mu(i)-v_\mu(j))}}{1-q^{\mu_i-\mu_j}t^{v_\mu(i)-v_\mu(j))}}.$$

□

22.5 Lecture 4: Proof of the symmetrization of E_μ

Proposition 22.5. *Let $\mu=(\mu_1,\dots,\mu_n)\in\mathbb{Z}^n$. Let $\lambda=(\lambda_1,\dots,\lambda_n)$ be the weakly decreasing rearrangement of μ and let $z_\mu\in S_n$ be minimal length such that $\mu=z_\mu\lambda$. Let*

$$W_\lambda=\{y\in S_n\mid y\lambda=\lambda\}\qquad\text{and}\qquad W_\lambda(t)=\sum_{y\in W_\lambda}t^{\ell(y)}.$$

Then

$$P_\lambda = \frac{t^{\frac12\ell(w_0)}}{W_\lambda(t)}\Big(\frac{1}{t^{\frac12\ell(z_\mu)}\mathrm{ev}^\rho_\lambda(c_{z_\mu}(Y))}\Big)\mathbf{1}_0E_\mu.$$

Alternatively,

$$P_\lambda = \frac{t^{\frac12\ell(w_0)}}{W_\lambda(t)}\Big(\prod_{(i,j)\in\mathrm{Inv}(z_\mu)}\frac{1-q^{\lambda_i-\lambda_j}t^{j-i}}{1-q^{\lambda_i-\lambda_j}t^{j-i+1}}\Big)\mathbf{1}_0E_\mu.$$

Proof. The proof is by induction on $\ell(z_\mu)$. The base case $z_\mu=1$ has $\mu=\lambda$ and $v_\lambda = w_0z_\lambda$ so that

$$\begin{aligned}F_\lambda = \mathbf{1}_0E_\lambda &= t^{-\frac12\ell(w_0)}\Big(\sum_{u\in S_n/S_\lambda}\sum_{v\in S_\lambda} t^{\frac12\ell(x)+\ell(y)}T_xT_y\Big)E_\lambda \\ &= t^{-\frac12\ell(w_0)}\Big(\sum_{u\in S_n/S_\lambda} t^{\frac12\ell(x)}T_x\Big)W_\lambda(t)E_\lambda = t^{-\frac12\ell(w_0)}W_\lambda(t)P_\lambda,\end{aligned}$$

where $T_yE_\lambda = t^{\frac12\ell(y)}E_y$ is a consequence of (Tigivest) and the last equality is (10.4). For the induction step, assume that μ is not weakly decreasing and let $i\in\{1,\ldots,n-1\}$ be such that $\mu_i<\mu_{i+1}$. Then $z_{s_i\mu}=s_iz_\mu$ and $\ell(z_{s_i\mu})=\ell(z_\mu)-1$. Using $E_\mu = t^{\frac12}\tau_i^\vee E_{s_i\mu}$ and $\mathbf{1}_0T_i=\mathbf{1}_0t^{\frac12}$ from (CXlambdaaction) and (Tigivest) gives

$$\begin{aligned}F_\mu = \mathbf{1}_0E_\mu = \mathbf{1}_0t^{\frac12}\tau_{i_1}E_{s_i\mu} &= \mathbf{1}_0\Big(t^{\frac12}T_i+\frac{1-t}{1-Y_i^{-1}Y_{i+1}}\Big)E_{s_i\mu} = \mathbf{1}_0\Big(t+\frac{1-t}{1-Y_i^{-1}Y_{i+1}}\Big)E_{s_i\mu}\\ = \mathbf{1}_0\frac{1-tY_i^{-1}Y_{i+1}}{1-Y_i^{-1}Y_{i+1}}E_{s_i\mu} &= \mathbf{1}_0\frac{1-tq^{\mu_{i+1}-\mu_i}t^{v_\mu(i+1)-v_\mu(i)}}{1-q^{\mu_{i+1}-\mu_i}t^{v_\mu(i+1)-v_\mu(i)}}E_{s_i\mu} = \frac{1-q^{\mu_{i+1}-\mu_i}t^{v_\mu(i+1)-v_\mu(i)+1}}{1-q^{\mu_{i+1}-\mu_i}t^{v_\mu(i+1)-v_\mu(i)}}F_{s_i\mu}\end{aligned}$$

and the result follows by induction. □

Here is a "better" proof:

Proof.

$$\mathbf{1}_0E_\mu = \mathbf{1}_0\tau^\vee_{z_\mu}E_\lambda = \mathbf{1}_0c_{z_\mu}(Y)E_\lambda = \mathrm{ev}^t_\lambda(c_{z_\mu}(Y))\mathbf{1}_0E_\lambda = \mathrm{ev}^t_\lambda(c_{z_\mu}(Y))t^{-\frac12\ell(w_0)}W_\lambda(t)P_\lambda,$$

□

22.6 Lecture 4: Proof of the KZ family properties

Proposition 22.6. *Let $\mu=(\mu_1,\ldots,\mu_n)\in\mathbb{Z}^n_{\ge0}$. Let $i\in\{1,\ldots,n-1\}$ and let T_i and g be as defined in (10.1). Then*

$$t^{\frac12}T_if_\mu = \begin{cases} f_{s_i\mu}, & \text{if } \mu_i>\mu_{i+1},\\ tf_\mu, & \text{if } \mu_i=\mu_{i+1},\end{cases}\qquad \text{and}\qquad gf_\mu = q^{-\mu_n}f_{(\mu_n,\mu_1,\ldots,\mu_{n-1})}.$$

Proof. Assume $\mu_i>\mu_{i+1}$. Then $z_{s_i\mu}=s_iz_\mu$ and $\ell(z_{s_i\mu})=\ell(z_\mu)+1$ so that

$$t^{\frac12}T_if_\mu = t^{\frac12}T_it^{\frac12\ell(z_\mu)}T_{z_\mu}E_\lambda = t^{\frac12\ell(z_{s_i\mu})}T_{z_{s_i\mu}}E_\lambda = f_{s_i\mu}.$$

Assume $\mu_i=\mu_{i+1}$. Then there exists $j\in\{1,\ldots,n-1\}$ such that $s_j\lambda=\lambda$ and $s_iz_\mu=z_\mu s_j$ (so that $s_i\mu=s_iz_\mu\lambda=z_\mu s_j\lambda$). Then

$$t^{\frac12}T_if_\mu = t^{\frac12}T_it^{\frac12\ell(z_\mu)}T_{z_\mu}E_\lambda = t^{\frac12\ell(z_\mu)}T_{z_\mu}t^{\frac12}T_jE_\lambda = t^{\frac12\ell(z_\mu)}T_{z_\mu}tE_\lambda = tf_\mu.$$

(c) Let $\mu = (\mu_1, \ldots, \mu_n)$ and let i and j be such that λ_i is the first part of λ equal to μ_n and λ_j is the last part of λ equal to μ_n. Thus $\mu_n = \lambda_i = \lambda_{i+1} = \cdots = \lambda_j$. Write $z_\mu = z s_{n-1} \cdots s_j$ with $z \in S_{n-1}$ and let $c_n = s_1 \cdots s_{n-1}$. Then, using that $v_\lambda(j) = 1 + (j-i) + n - j = n - i + 1$ gives

$$\begin{aligned}
gf_\mu &= gt^{\frac{1}{2}\ell(z_\mu)} T_{z_\mu} E_\lambda = gt^{\frac{1}{2}\ell(z)} T_z t^{\frac{1}{2}(n-j)} T_{n-1} \cdots T_j E_\lambda = t^{\frac{1}{2}(n-j)} g t^{\frac{1}{2}\ell(z)} T_z g^{-1} g T_{n-1} \cdots T_j E_\lambda \\
&= t^{\frac{1}{2}(n-j)} (g t^{\frac{1}{2}\ell(z)} T_z g^{-1}) T_1 \cdots T_{j-1} (T_{j-1}^{-1} \cdots T_1^{-1} g T_{n-1} \cdots T_j) E_\lambda \\
&= t^{\frac{1}{2}(n-j)} (t^{\frac{1}{2}\ell(z)} T_{c_n z c_n^{-1}}) T_1 \cdots T_{j-1} Y_j E_\lambda \\
&= t^{\frac{1}{2}(n-j)} (t^{\frac{1}{2}\ell(z)} T_{c_n z c_n^{-1}}) T_1 \cdots T_{j-1} q^{-\lambda_j} t^{-(v_\lambda(j)-1)+\frac{1}{2}(n-1)} E_\lambda \\
&= q^{-\lambda_j} t^{\frac{1}{2}(n-j)-(n-i+1-1)+\frac{1}{2}(n-1)} (t^{\frac{1}{2}\ell(z)} T_{c_n z c_n^{-1}}) T_1 \cdots T_{i-1} T_i \cdots T_{j-1} E_\lambda \\
&= q^{-\mu_n} t^{-\frac{1}{2}j+i-\frac{1}{2}} (t^{\frac{1}{2}\ell(z)} T_{c_n z c_n^{-1}}) T_1 \cdots T_{i-1} t^{\frac{1}{2}(j-i)} E_\lambda \\
&= q^{-\mu_n} (t^{\frac{1}{2}\ell(z)} T_{c_n z c_n^{-1}}) t^{\frac{1}{2}(i-1)} T_1 \cdots T_{i-1} E_\lambda = q^{-\mu_n} f_{(\lambda_i, \mu_1, \ldots, \mu_{n-1})} = q^{-\mu_n} f_{(\mu_n, \mu_1, \ldots, \mu_{n-1})},
\end{aligned}$$

where the next to last equality follows from $s_1 \cdots s_{i-1}(\lambda_1, \ldots, \lambda_n) = (\lambda_i, \lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n)$ and $c_n z c_n^{-1}(\lambda_i, \lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n) = (\lambda_i, \mu_1, \ldots, \mu_{n-1})$. □