24 Lecture 5: Proofs of Principal specialization formulas

24.1 Lecture 5: Proof of the c-function formula

Theorem 24.1. *Let $\mu, \lambda \in \mathbb{Z}^n$ with $\lambda_1 \ge \cdots \ge \lambda_n$ and let $E_\mu(x_1, \ldots, x_n; q, t)$ and $P_\lambda(x_1, \ldots, x_n; q, t)$ and $A_{\lambda+\rho}(x_1, \ldots, x_n; q, t)$ be the corresponding nonsymmetric, symmetric and fermionic Macdonald polynomials, respectively. Then*

$$\begin{aligned} E_\mu(1, t, t^2, \ldots, t^{n-1}; q, t) &= t^{\frac{(n-1)}{2}|\lambda|} t^{-\frac{1}{2}\ell(v_\mu^{-1})} \mathrm{ev}_0^t(c_{u_\mu}(Y^{-1})), \\ P_\lambda(1, t, t^2, \ldots, t^{n-1}; q, t) &= t^{\frac{(n-1)}{2}|\lambda|} \mathrm{ev}_0^{t^{-1}}(c_{t_\lambda}(Y^{-1})) \qquad \text{and} \\ A_{\lambda+\rho}(1, t, t^2, \ldots, t^{n-1}; q, t) &= 0, \end{aligned}$$

where $|\mu| = \mu_1 + \cdots + \mu_n$.

Proof. For this proof use the realization of the polynomial representation $\mathbb{C}[X]$ as an induced module $\widetilde{H}\mathbf{1}_Y$ via the $\widetilde{H}$-module isomorphism of (CXasIndHY). Then the creation formulas for E_μ, P_λ and $A_{\lambda+\rho}$ are

$$E_\mu = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee \mathbf{1}_0, \qquad P_\lambda = \frac{t^{\frac{1}{2}\ell(w_0)}}{W_\lambda(t)} \mathbf{1}_0 E_\lambda, \qquad \text{and} \qquad A_{\lambda+\rho} = t^{\frac{1}{2}\ell(w_0)} \varepsilon_0 E_{\lambda+\rho}$$

(see Theorem 2.6 and (creationPA)).

Let $\mathbf{1}_X$ be a formal symbol which satisfies $\mathbf{1}_X T_j = t^{\frac{1}{2}} \mathbf{1}_X$ and $\mathbf{1}_X g^\vee = \mathbf{1}_X$, for $j \in \{1, \ldots, n-1\}$. Since $g^\vee = x_1 T_1 \cdots T_{n-1}$ and $x_{i+1} = T_i x_i T_i$ then $x_1 = g^\vee T_{n-1}^{-1} \cdots T_1^{-1}$ and

$$\mathbf{1}_X x_i = t^{-\frac{1}{2}(n-1)} t^{i-1} \mathbf{1}_X, \qquad \text{for } i \in \{1, \ldots, n\}.$$

Thus, if $\mu \in \mathbb{Z}^n$ then

$$\mathbf{1}_X E_\mu(x_1, \ldots, x_n; q, t) = \mathbf{1}_X t^{-\frac{1}{2}(n-1)|\mu|} E_\mu(1, t, t^2, \ldots, t^{n-1}; q, t), \qquad \text{where} \quad |\mu| = \mu_1 + \cdots + \mu_n.$$

For $i \in \{1, \ldots, n-1\}$,

$$\begin{aligned} \mathbf{1}_X \tau_i^\vee &= \mathbf{1}_X \Big(T_i + \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}{1 - Y_i^{-1} Y_{i+1}} \Big) = \mathbf{1}_X \Big(t^{\frac{1}{2}} + \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}{1 - Y_i^{-1} Y_{i+1}} \Big) = \mathbf{1}_X \Big(\frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} Y_i^{-1} Y_{i+1}}{1 - Y_i^{-1} Y_{i+1}} \Big) \\ &= \mathbf{1}_X c_{i,i+1}(Y^{-1}), \end{aligned}$$

By (EeigenvalueB),

$$c_{i,i+1}(Y^{-1}) \mathbf{1}_Y = \mathrm{ev}_0^t(c_{i,i+1}(Y^{-1})) \mathbf{1}_Y.$$

If $w \in W$ and $\ell(s_i w) > \ell(w)$ then

$$\mathbf{1}_X \tau_i^\vee \tau_w^\vee \mathbf{1}_Y = \mathbf{1}_X c_{\alpha_i^\vee}(Y^{-1}) \tau_w^\vee \mathbf{1}_Y = \mathbf{1}_X \tau_w^\vee c_{w^{-1}\alpha_i^\vee}(Y^{-1}) \mathbf{1}_Y = \mathrm{ev}_0^t(c_{w^{-1}\alpha_i^\vee}(Y^{-1})) \mathbf{1}_X \tau_w^\vee \mathbf{1}_Y.$$

This is the induction step to conclude that if $w \in W$ and $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word for w then

$$\mathbf{1}_X \tau_w^\vee \mathbf{1}_Y = \mathbf{1}_X \tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee \mathbf{1}_Y = \mathbf{1}_X \mathrm{ev}_{k\rho}(c_w(Y^{-1}) \mathbf{1}_Y = \mathrm{ev}_0^t(c_w(Y^{-1})) \mathbf{1}_X \mathbf{1}_Y.$$

Thus, by the creation formula $E_\mu = t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee \mathbf{1}_Y$,

$$\begin{aligned} t^{-\frac{1}{2}(n-1)|\mu|} E_\mu(1, t, \ldots, t^{n-1}; q, t) \mathbf{1}_X \mathbf{1}_Y = \mathbf{1}_X E_\mu \mathbf{1}_Y &= \mathbf{1}_X t^{-\frac{1}{2}\ell(v_\mu^{-1})} \tau_{u_\mu}^\vee \mathbf{1}_Y \\ &= t^{-\frac{1}{2}\ell(v_\mu^{-1})} \mathrm{ev}_0^t(c_{u_\mu}(Y^{-1})) \mathbf{1}_X \mathbf{1}_Y. \end{aligned}$$

which completes the proof of the first statement.

Using the creation formula $A_{\lambda+\rho} = t^{\frac{1}{2}\ell(w_0)}\varepsilon_0 E_{\lambda+\rho}$ gives

$$t^{-\frac{1}{2}(n-1)|\lambda|}A_{\lambda+\rho}(1,t,\ldots,t^{n-1};q,t)\mathbf{1}_X\mathbf{1}_Y = \mathbf{1}_X A_{\lambda+\rho}\mathbf{1}_Y = t^{\frac{1}{2}\ell(w_0}\mathbf{1}_X\varepsilon_0 E_{\lambda+\rho}\mathbf{1}_Y = 0,$$

since $\mathbf{1}_X\varepsilon_0 = 0$. Thus $A_{\lambda+\rho}(1,t,\ldots,t^{n-1};q,t) = 0$.

Using the creation formula $P_\lambda = \frac{t^{\frac{1}{2}\ell(w_0)}}{W_\lambda(t)}\mathbf{1}_0 E_\lambda$ gives

$$\begin{aligned} t^{-\frac{1}{2}(n-1)|\lambda|}P_\lambda(1,t,\ldots,t^{n-1};q,t)\mathbf{1}_X\mathbf{1}_Y &= \mathbf{1}_X P_\lambda \mathbf{1}_Y = \mathbf{1}_X\frac{t^{\frac{1}{2}\ell(w_0)}}{W_\lambda(t)}\mathbf{1}_0 E_\lambda\mathbf{1}_Y \\ &= \mathbf{1}_X\frac{t^{\frac{1}{2}\ell(w_0)}W_0(t)}{W_\lambda(t)}E_\lambda\mathbf{1}_Y = t^{\frac{1}{2}\ell(w_0)}t^{-\frac{1}{2}\ell(v_\lambda^{-1})}\frac{W_0(t)}{W_\lambda(t)}\mathrm{ev}_{k\rho}(c_{u_\lambda}(Y^{-1}))\mathbf{1}_X\mathbf{1}_Y. \end{aligned}$$

Since $v_\lambda^{-1} = (w_0w_\lambda)^{-1} = w_\lambda w_0$ and $t_\lambda = u_\lambda v_\lambda$ then

$$\begin{aligned} \frac{t^{-\frac{1}{2}\ell(w_0)}W_0(t)}{t^{-\frac{1}{2}\ell(w_\lambda)}W_\lambda(t)}\mathrm{ev}_0^t(c_{u_\lambda}(Y^{-1})) &= \mathrm{ev}_0^{t^{-1}}\Big(\frac{c_{w_0}(Y))}{c_{w_\lambda}(Y)}\Big)\mathrm{ev}_0^t(c_{u_\lambda}(Y^{-1})) \\ &= \mathrm{ev}_0^{t^{-1}}\Big(\frac{c_{w_0}(Y^{-1}))}{c_{w_\lambda}(Y^{-1})}\Big)\mathrm{ev}_0^{t^{-1}}(v_\lambda^{-1}c_{u_\lambda}(Y^{-1})) = \mathrm{ev}_0^{t^{-1}}(c_{v_\lambda}(Y^{-1}))\mathrm{ev}_0^{t^{-1}}(v_\lambda^{-1}c_{u_\lambda}(Y^{-1})) \\ &= \mathrm{ev}_0^{t^{-1}}(c_{u_\lambda v_\lambda}(Y^{-1})) = \mathrm{ev}_0^{t^{-1}}(c_{t_\lambda}(Y^{-1})), \end{aligned}$$

where the first equality comes from (PoinbysymmB) which gives

$$t^{-\frac{1}{2}\ell(w_0)}W_0(t) = \mathrm{ev}_0^t(c_{w_0}(Y)) = \mathrm{ev}_0^{t^{-1}}(c_{w_0}(Y^{-1})),$$

and the second equality results from the fact that

$$\mathrm{ev}_0^t(c_{u_\lambda}(Y^{-1})) = \mathrm{ev}_0^{t^{-1}}(v_\lambda^{-1}c_{u_\lambda}(Y^{-1})), \qquad \text{since } \mathrm{ht}(\alpha^\vee) = \mathrm{ht}(-w_0\alpha^\vee) \text{ for } \alpha^\vee \in \mathrm{Inv}(w_0).$$

□

24.2 Lecture 5: Proof of the root product formula

Corollary 24.2. *Let $\mu \in \mathbb{Z}_{\geq 0}^n$ and let λ be the decreasing rearrangement of μ. Let $n(\lambda) = \sum_{i=1}^n (i-1)\lambda_i$. Then*

$$P_\lambda(1,t,t^2,\ldots,t^{n-1};q,t) = t^{n(\lambda)}\prod_{1\le i<j\le n}\prod_{\ell=0}^{\lambda_i-\lambda_j-1}\frac{1-q^\ell t^{j-i+1}}{1-q^\ell t^{j-i}}$$

and

$$E_\mu(1,t,t^2,\ldots,t^{n-1};q,t) = t^{-\frac{1}{2}\ell(v_\mu^{-1})}\prod_{(r,c)\in\mu}\prod_{i=1}^{u_\mu(r,c)}\frac{1-q^{\mu_r-c+1}t^{v_\mu(r)-i+1}}{1-q^{\mu_r-c+1}t^{v_\mu(r)-i}}.$$

Proof. Let $\rho^\vee = \frac{1}{2}(n-1, n-3, \ldots, -(n-3), -(n-1))$. Then

$$n(\lambda) = \langle(\lambda_1,\ldots,\lambda_n),(0,1,\ldots,n-1)\rangle = \langle\lambda, -\rho^\vee + \tfrac{(n-1)}{2}(1,1,\ldots,1)\ \rangle = \tfrac{(n-1)}{2}|\lambda| - \langle\lambda,\rho^\vee\rangle$$

and, since $\ell(t_\lambda) = \sum_{i<j}(\lambda_i - \lambda_j)$ then

$$\begin{aligned} \ell(t_\lambda) &= (n-1)\lambda_1 + (n-2)\lambda_2 + \cdots + (n-n)\lambda_n - \big((n-1)\lambda_n + (n-2)\lambda_{n-1} + \cdots + (n-n)\lambda_1\big) \\ &= (n-1)\lambda_1 + (n-3)\lambda_2 + (n-5)\lambda_3 + \cdots - (n-3)\lambda_{n-1} - (n-1)\lambda_n = \langle\lambda, 2\rho^\vee\rangle, \end{aligned}$$

so that

$$n(\lambda) = \tfrac{1}{2}(n-1)|\lambda| - \tfrac{1}{2}\ell(t_\lambda). \qquad \text{(nlambda)}$$

Since

$$\begin{aligned}\mathrm{ev}_0^{t^{-1}}(c_{i,j+rn}(Y^{-1})) &= \mathrm{ev}_0^{t^{-1}}\Big(t^{-\frac12}\frac{1-tq^rY_i^{-1}Y_j}{1-q^rY_i^{-1}Y_j}\Big) \\ &= t^{-\frac12}\frac{1-tq^rt^{-(i-1)+\frac12(n-1)}t^{(j-1)-\frac12(n-1)}}{1-q^rt^{-(i-1)+\frac12(n-1)}t^{(j-1)-\frac12(n-1)}} = t^{-\frac12}\frac{1-q^rt^{j-i+1}}{1-q^rt^{j-i}}\end{aligned}$$

then, using (Invfortlambda) and (nlambda),

$$t^{\frac{(n-1)}{2}|\lambda|}\mathrm{ev}_0^{t^{-1}}(c_{t_\lambda}(Y^{-1})) = t^{\frac{(n-1)}{2}|\lambda|}t^{-\frac12\ell(t_\lambda)}\prod_{i<j}\prod_{r=0}^{\lambda_i-\lambda_j-1}\frac{1-q^rt^{j-i+1}}{1-q^rt^{j-i}} = t^{n(\lambda)}\prod_{i<j}\prod_{r=0}^{\lambda_i-\lambda_j-1}\frac{1-q^rt^{j-i+1}}{1-q^rt^{j-i}},$$

which, by Theorem 5.1, gives the formula in the first statement.

By Theorem 5.1,

$$\begin{aligned}E_\mu(1,t,t^2,\ldots,t^{n-1};q,t) &= t^{\frac{(n-1)}{2}|\mu|}t^{-\frac12\ell(v_\mu^{-1})}\mathrm{ev}_{k\rho}(c_{u_\mu}(Y^{-1})) \\ &= t^{\frac{(n-1)}{2}|\mu|-\frac12\ell(u_\mu)-\frac12\ell(v_\mu)}\Big(t^{\frac12\ell(u_\mu)}\mathrm{ev}_{k\rho}(c_{u_\mu}(Y^{-1}))\Big)\end{aligned}$$

and, if λ is the weakly decreasing rearrangement of μ then, by (nlambda),

$$\tfrac{(n-1)}{2}|\mu| - \tfrac12\ell(u_\mu) - \tfrac12\ell(v_\mu) = \tfrac{(n-1)}{2}|\lambda| - \tfrac12\ell(t_\mu) = \tfrac{(n-1)}{2}|\lambda| - \tfrac12\ell(t_\lambda) = n(\lambda).$$

Then (Invforumu) gives that $t^{\frac12\ell(u_\mu)}\mathrm{ev}_0^t(c_{u_\mu}(Y^{-1}))$ is equal to

$$\mathrm{ev}_0^t\Big(\prod_{b=(r,c)\in\mu}\prod_{j=1}^{u_\mu(r,c)}\frac{1-tY^{-((\mu_r-c+1)K+\alpha_{v_\mu(r),j})}}{1-Y^{-((\mu_r-c+1)K+\alpha_{v_\mu(r),j})}}\Big) = \prod_{b=(r,c)\in\mu}\prod_{j=1}^{u_\mu(r,c)}\frac{1-tq^{\mu_r-c+1}t^{v_\mu(r)-j}}{1-q^{\mu_r-c+1}t^{v_\mu(r)-j}}.$$

□

24.3 Lecture 5: Proof of the hook formula

Theorem 24.3. *Let $\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{Z}_{\geq 0}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$. Then*

$$P_\lambda(1,t,t^2,\ldots,t^{n-1};q,t) = t^{n(\lambda)}\prod_{b\in\lambda}\frac{1-q^{\mathrm{coarm}_\lambda(b)}t^{n-\mathrm{coleg}_\lambda(b)}}{1-q^{\mathrm{arm}_\lambda(b)}t^{\mathrm{leg}_\lambda(b)+1}}.$$

Proof. In view of Corollary 5.2 the result will follow if we prove that

$$\prod_{i<j}\prod_{\ell=0}^{\lambda_i-\lambda_j-1}\frac{1-q^\ell t^{j-i+1}}{1-q^\ell t^{j-i}} = \prod_{b\in\lambda}\frac{1-q^{\mathrm{coarm}_\lambda(b)}t^{n-\mathrm{coleg}_\lambda(b)}}{1-q^{\mathrm{arm}_\lambda(b)}t^{\mathrm{leg}_\lambda(b)+1}}.$$

The left hand side is

$$LHS = \prod_{i<j}\prod_{\ell=0}^{\lambda_i-\lambda_j-1}\frac{1-q^\ell t^{j-i+1}}{1-q^\ell t^{j-i}} = \prod_{r=1}^{n}\prod_{j=r+1}^{n}\prod_{\ell=0}^{\lambda_r-\lambda_j-1}\frac{1-tq^\ell t^{j-r}}{1-q^\ell t^{j-r}}.$$

Let m be the number of columns of length n in λ. Let $r\in\{1,\ldots,n\}$. Switching the products over j and ℓ gives

$$\prod_{j=r+1}^{n}\prod_{\ell=0}^{\lambda_r-\lambda_j-1}\frac{1-tq^\ell t^{j-r}}{1-q^\ell t^{j-r}}=\prod_{c=m+1}^{\lambda_r}\prod_{j=\lambda'_c+1}^{n}\frac{1-tq^{\lambda_r-c}t^{j-r}}{1-q^{\lambda_r-c}t^{j-r}}=\prod_{c=m+1}^{\lambda_r}\frac{1-tq^{\lambda_r-c}t^{n-r}}{1-q^{\lambda_r-c}t^{\lambda'_c+1-r}} \qquad (*)$$

The definitions of arms, legs, coarms and colegs of boxes give that

$$RHS=t^{n(\lambda)}\prod_{b=(r,c)\in\lambda}\frac{1-q^{\mathrm{coarm}_\lambda(b)}t^{n-\mathrm{coleg}_\lambda(b)}}{1-q^{\mathrm{arm}_\lambda(b)}t^{\mathrm{leg}_\lambda(b)+1}}=t^{n(\lambda)}\prod_{r=1}^{n}\prod_{c=1}^{\lambda_r}\frac{1-q^{c-1}t^{n-(r-1)}}{1-q^{\lambda_r-c}t^{\lambda'_c-r+1}}$$

For $r\in\{1,\ldots,n\}$ let $\ell=\lambda_r$ and write

$$\prod_{c=1}^{\lambda_r}\frac{1-q^{c-1}t^{n-(r-1)}}{1-q^{\lambda_r-c}t^{\lambda'_c-r+1}}=\frac{(1-q^0t^{n-(r-1)})(1-q^1t^{n-(r-1)})\cdots(1-q^{\lambda_r-1}t^{n-(r-1)})}{(1-q^{\lambda_r-1}t^{\lambda'_1-r+1})\cdots(1-q^1t^{\lambda'_{\ell-1}-r-1+1})(1-q^0t^{\lambda'_\ell-r+1})}$$

to observe that the last m factors in the numerator cancel with the first m terms in the denominator. Thus

$$\begin{aligned}\prod_{c=1}^{\lambda_r}\frac{1-q^{c-1}t^{n-(r-1)}}{1-q^{\lambda_r-c}t^{\lambda'_c-r+1}}&=\frac{(1-q^0t^{n-(r-1)})(1-q^1t^{n-(r-1)})\cdots(1-q^{\lambda_r-m}t^{n-(r-1)})}{(1-q^{\lambda_r-m}t^{\lambda'_{\ell-m}-r+1})\cdots(1-q^1t^{\lambda'_{\ell-1}-r-1+1})(1-q^0t^{\lambda'_\ell-r+1})}\\&=\prod_{c=m+1}^{\lambda_r}\frac{1-q^{\lambda_r-c}t^{n-(r-1)}}{1-q^{\lambda-c}t^{\lambda'_c-r+1}}=\prod_{c=m+1}^{\lambda_r}\frac{1-tq^{\lambda_r-c}t^{n-r}}{1-q^{\lambda-c}t^{\lambda'_c-r+1}}.\end{aligned}$$

Since this is equal to the expression in (*) the result follows. □

24.4 Lecture 5: Proof of the nonsymmetric hook formula

Theorem 24.4. *Let $\mu=(\mu_1,\ldots,\mu_n)\in\mathbb{Z}^n_{\ge0}$ and let λ be the weakly decreasing rearrangment of μ. For $r\in\{1,\ldots,n\}$ and $c\in\{1,\ldots,\mu_r\}$ define*

$$u_\mu(r,c)=\#\{r'\in\{1,\ldots,r-1\}\mid \mu_{r'}<c\le\mu_r\}+\#\{r'\in\{r+1,\ldots,n\}\mid\mu_{r'}<c-1<\mu_r\})\quad\text{and}$$
$$v_\mu(r)=1+\#\{r'\in\{1,\ldots,r-1\}\mid\mu_{r'}\le\mu_r\}+\#\{r'\in\{r+1,\ldots,n\}\mid\mu_{r'}<\mu_r\}.$$

Let $n(\lambda)=\sum_{i=1}^n(i-1)\lambda_i$. Then

$$E_\mu(1,t,t^2,\ldots,t^{n-1};q,t)=t^{n(\lambda)}\prod_{(r,c)\in\mu}\frac{1-q^ct^{v_\mu(r)}}{1-q^{\mu_r-c+1}t^{v_\mu(r)-u_\mu(r,c)}}.$$

Proof. In view of Corollary 5.2 the result will follow if we prove that

$$\prod_{b=(r,c)\in\mu}\prod_{j=1}^{u_\mu(r,c)}\frac{1-q^{\mu_r-c+1}t^{v_\mu(r)-j+1}}{1-q^{\mu_r-c+1}t^{v_\mu(r)-j}}=\prod_{(r,c)\in\mu}\frac{1-q^ct^{v_\mu(r)}}{1-q^{\mu_r-c+1}t^{v_\mu(r)-u_\mu(r,c)}}.$$

For a single box $b=(r,c)$, the product $\displaystyle\prod_{j=1}^{u_\mu(r,c)}\frac{1-tq^{\mu_r-c+1}t^{v_\mu(r)-j}}{1-q^{\mu_r-c+1}t^{v_\mu(r)-j}}$ is equal to

$$\begin{aligned}&\frac{(1-q^{\mu_r-c+1}t^{v_\mu(r)})}{(1-q^{\mu_r-c+1}t^{v_\mu(r)-1})}\frac{(1-q^{\mu_r-c+1}t^{v_\mu(r)-1})}{(1-q^{\mu_r-c+1}t^{v_\mu(r)-2})}\cdots\frac{(1-q^{\mu_r-c+1}t^{v_\mu(r)-(u_\mu(r,c)-1)})}{(1-q^{\mu_r-c+1}t^{v_\mu(r)-u_\mu(r,c)})}\\&=\frac{1-q^{\mu_r-c+1}t^{v_\mu(r)}}{1-q^{\mu_r-c+1}t^{v_\mu(r)-u_\mu(r,c)}}.\end{aligned}$$

For a single fixed row r, the product $\displaystyle\prod_{c=1}^{\mu_r}(1-q^{\mu_r-c+1}t^{v_\mu(r)}) = \prod_{c=1}^{\mu_r} 1-q^c t^{v_\mu(r)}$, and so

$$\prod_{(r,c)\in\mu}\prod_{j=1}^{u_\mu(r,c)} \frac{1-q^{\mu_r-c+1}t^{v_\mu(r)-j+1}}{1-q^{\mu_r-c+1}t^{v_\mu(r)-j}} = \prod_{(r,c)\in\mu}\frac{1-q^{\mu_r-c+1}t^{v_\mu(r)}}{1-q^{\mu_r-c+1}t^{v_\mu(r)-u_\mu(r,c)}} = \prod_{(r,c)\in\mu}\frac{1-q^c t^{v_\mu(r)}}{1-q^{\mu_r-c+1}t^{v_\mu(r)-u_\mu(r,c)}}.$$

□