8 Lecture 7: Proofs

8.1 Lecture 7: Proof of the Boson Fermion equalities

Proposition 8.1. *With notations as in (BosFermmaps), (symms) and (bosfersymm),*

$$p_0\mathbb{C}[X] = \mathbb{C}[X]^{W_0}, \qquad e_0\mathbb{C}[X] = \mathbb{C}[X]^{\mathrm{det}} = a_\rho\mathbb{C}[X]^{W_0} \quad \textit{and} \quad a_\rho = e_0x^\rho,$$

$$\mathbf{1}_0\mathbb{C}[X] = \mathbb{C}[X]^{\mathrm{Bos}} = \mathbb{C}[X[^{W_0}, \qquad \varepsilon_0\mathbb{C}[X] = \mathbb{C}[X]^{\mathrm{Fer}} = A_\rho\mathbb{C}[X]^{W_0} \quad \textit{and} \quad A_\rho = \varepsilon_0x^\rho.$$

Proof. Recall that

$$p_0 = \sum_{w\in S_n} w.$$

(1a) If $f \in \mathbb{C}[X]^{S_n}$ then $f = p_0\left(\frac{1}{n!}f\right)$ so that $f \in p_0\mathbb{C}[X]$. So $\mathbb{C}[X]^{S_n} \subseteq p_0\mathbb{C}[X]$.

(1b) Assume $f \in p_0\mathbb{C}[X]$. Then $f = p_0g$ and if $w \in S_n$ then $wf = wp_0g = p_0g$, and so $f \in \mathbb{C}[X]^{S_n}$. So $p_0\mathbb{C}[X] \subseteq \mathbb{C}[X]^{S_n}$.

Combining (1a) and (1b) gives $p_0\mathbb{C}[X] = \mathbb{C}[X]^{S_n}$.

(1c) Let $s_{ij} \in S_n$ denote the transposition that switches i and j.
Assume $f \in \mathbb{C}[X]^{\mathrm{det}}$. Then $(1 - s_{ij})f = 0$ and so f is divisible by $x_j - x_i$. So

$$f \text{ is divisible by} \quad a_\rho = \prod_{1\le i<j\le n} x_j - x_i.$$

Then $\frac{1}{a_\rho}f \in \mathbb{C}[X]^{S_n}$. So $f \in a_\rho\mathbb{C}[X]^{W_0}$. So $\mathbb{C}[X]^{\mathrm{det}} \subseteq a_\rho\mathbb{C}[X]^{S_n}$.

(1d) Assume $f \in a_\rho\mathbb{C}[X]^{S_n}$ and let $g \in \mathbb{C}[X]^{S_n}$ be such that $f = a_\rho g$.
Then $s_{ij}f = (s_{ij}a_\rho)(s_{ij}g) = -a_\rho g = -f$. So $f \in \mathbb{C}[X]^{\mathrm{det}}$. Thus $a_\rho\mathbb{C}[X]^{W_0} \subseteq \mathbb{C}[X]^{\mathrm{det}}$.

Combining (1c) and (1d) gives $a_\rho\mathbb{C}[X]^{W_0} = \mathbb{C}[X]^{\mathrm{det}}$.

Recall that

$$e_0 = \sum_{w\in S_n} (-1)^{\ell(w)}w.$$

(1e) If $f \in e_0\mathbb{C}[X]$ then $s_\alpha f = s_\alpha e_0g = -e_0g = -f$. So $f \in \mathbb{C}[X]^{\mathrm{det}}$. So $e_0\mathbb{C}[X] \subseteq \mathbb{C}[X]^{\mathrm{det}}$.

(1f) If $f \in \mathbb{C}[X]^{\mathrm{det}}$ then $e_0f = \mathrm{Card}(W_0)f$. So $f \in e_0\mathbb{C}[X]$. So $\mathbb{C}[X]^{\mathrm{det}} \subseteq e_0\mathbb{C}[X]$.

Combining (1e) and (1f) gives $e_0\mathbb{C}[X] = \mathbb{C}[X]^{\mathrm{det}}$.

Since $e_0x^\rho \in e_0\mathbb{C}[X] \subseteq a_\rho\mathbb{C}[X]^{W_0}$ and the top coefficient of e_0x^ρ is x^ρ, which is the same as the top coefficient of a_ρ. Hence $e_0x^\rho = a_\rho$.

Recall that

$$\mathbf{1}_0 = \sum_{z\in S_n} t^{\frac{1}{2}(\ell(z)-\ell(w_0))}T_z.$$

(2a) Show that $\mathbf{1}_0\mathbb{C}[X] \subseteq \mathbb{C}[X]^{\mathrm{Bos}}$: Let $h \in \mathbf{1_0}\mathbb{C}[X]$. Write $h = \mathbf{1}_0f$ with $f \in \mathbb{C}[X]$. Then

$$T_{s_i}h = T_{s_i}\mathbf{1}_0f = t^{\frac{1}{2}}\mathbf{1}_0f = t^{\frac{1}{2}}h. \qquad \text{So } h \in \mathbb{C}[X]^{\mathrm{Bos}} \text{ and } \mathbf{1}_0\mathbb{C}[X] \subseteq \mathbb{C}[X]^{\mathrm{Bos}}.$$

(2b) Show that $\mathbb{C}[X]^{\mathrm{Bos}} \subseteq \mathbb{C}[X]^{W_0}$: Let $f \in \mathbb{C}[X]^{\mathrm{Bos}}$. Then, by Proposition 4.1 and (Poinbysymm),

$$f = \frac{t^{\frac{1}{2}\ell(w_0}}{W_0(t)}\mathbf{1}_0f = \frac{1}{[n]!}\sum_{w\in W_0} w\Big(f \prod_{1\le i<j\le n}\frac{x_i - tx_j}{x_i - x_j}\Big) \quad \in \mathbb{C}[X]^{W_0}. \qquad \text{So } \mathbb{C}[X]^{\mathrm{Bos}} \subseteq \mathbb{C}[X]^{W_0}.$$

(2c) Show that $\mathbb{C}[X]^{W_0} \subseteq \mathbf{1}_0\mathbb{C}[X]$: Assume $f \in \mathbb{C}[X]^{W_0}$. Then, by Proposition 4.1 and (Poinbysymm),

$$\mathbf{1}_0 \frac{t^{\frac{1}{2}\ell(w_0}}{W_0(t)} f = \frac{1}{[n]!}\sum_{w\in W_0} w\Big(f \prod_{1\le i<j\le n} \frac{x_i - tx_j}{x_i - x_j}\Big) = f\frac{1}{[n]!}\sum_{w\in W_0} w\Big(\prod_{1\le i<j\le n} \frac{x_i - tx_j}{x_i - x_j}\Big) = f.$$

So $f \in \mathbf{1}_0\mathbb{C}[X]$. Thus, $\mathbb{C}[X]^{W_0} \subseteq \mathbf{1}_0\mathbb{C}[X]$.

Combining (2a), (2b) and (2c) gives $\mathbf{1}_0\mathbb{C}[X] = \mathbb{C}[X]^{\mathrm{Bos}} = \mathbb{C}[X]^{S_n}$.

Recall that

$$\varepsilon_0 = \sum_{w\in S_n} (-t^{-\frac{1}{2}})^{\ell(z)-\ell(w_0)} T_z.$$

(2d) Show that $\varepsilon_0\mathbb{C}[X] \subseteq \mathbb{C}[X]^{\mathrm{Fer}}$: Assume $h = \varepsilon_0\mathbb{C}[X]$ and let $f \in \mathbb{C}[X]$ such that $h = \varepsilon_0 f$. Then

$$T_{s_i}h = T_{s_i}\varepsilon_0 f = -t^{-\frac{1}{2}}\varepsilon_0 f = -t^{-\frac{1}{2}}h. \qquad \text{So } h \in \mathbb{C}[X]^{\mathrm{Fer}} \text{ and } \varepsilon_0\mathbb{C}[X] \subseteq \mathbb{C}[X]^{\mathrm{Fer}}.$$

(2e) Show that $\mathbb{C}[X]^{Fer} \subseteq A_\rho\mathbb{C}[X]^{W_0}$: Let $f \in \mathbb{C}[X]^{\mathrm{Fer}}$. Then $T_i f = -t^{-\frac{1}{2}}f$ gives

$$f = \frac{t^{\frac{1}{2}\ell(w_0)}}{W_0(t)}\varepsilon_0 f = \frac{1}{W_0(t)}\frac{A_\rho}{a_\rho}\sum_{w\in W_0}\det(w)wf \quad \in A_\rho\mathbb{C}[X]^{W_0}.$$

So $\mathbb{C}[X]^{\mathrm{Fer}} \subseteq A_\rho\mathbb{C}[X]^{W_0}$.

(2f) Show that $A_\rho\mathbb{C}[X]^{W_0} \subseteq \mathbb{C}[X]^{Fer}$: Assume $A_\rho\mathbb{C}[X]^{W_0}$. Let $g \in \mathbb{C}[X]^{S_n}$ be such that $f = A_\rho g$ and write g as a linear combination, $g = \sum c_\lambda s_\lambda$, where s_λ are Schur functions. Then

$$f = A_\rho g = \sum_\lambda c_\lambda A_\rho s_\lambda = \sum_\lambda c_\lambda \frac{A_\rho}{a_\rho}\sum_{w\in W_0}\det(w_0 w) w x^{\lambda+\rho} = \varepsilon_0\Big(\sum_\lambda c_\lambda x^{\lambda+\rho}\Big),$$

by (slicksymmA) and the fact that

$$t^{-\frac{1}{2}\ell(w_0)}\frac{A_\rho}{a_\rho} = t^{-\frac{1}{2}\ell(w_0)}\prod_{1\le i<j\le n}\frac{x_j - tx_i}{x_j - x_i} = \prod_{1\le i<j\le n}\frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}x_i x_j^{-1}}{1 - x_i x_j^{-1}} = c_{w_0}(x).$$

So $f \in \varepsilon_0\mathbb{C}[X]$. Thus, $A_\rho\mathbb{C}[X]^{W_0} \subseteq \varepsilon_0\mathbb{C}[X]$.

Combining (2d), (2e) and (2f) gives $\varepsilon_0\mathbb{C}[X] = \mathbb{C}[X]^{\mathrm{Fer}} = A_\rho\mathbb{C}[X]^{W_0}$. □

8.2 Lecture 7: Proof that $(,)_{q,t}$ is normalized Hermitian and nondegenerate

Proposition 8.2.

(a) (sesquilinear) If $f, g \in \mathbb{C}[X]$ and $c \in \mathbb{C}[q^{\pm 1}]$ then

$$(cf, g)_{q,t} = c(f,g)_{q,t}, \qquad \text{and} \qquad (f, cg)_{q,t} = \overline{c}(f,g)_{q,t}.$$

(b) (nonisotropy) If $f \in \mathbb{C}[X]$ and $f \ne 0$ then $(f,f)_{q,t} \ne 0$.

(c) (nondegeneracy) If F is a subspace of $\mathbb{C}[X]$ and $(,)_F\colon F\times F \to \mathbb{C}$ is the restriction of $(,)_{q,t}$ to F, then $(,)_F$ is nondegenerate.

(d) (normalized Hermitian) If $f_1, f_2 \in \mathbb{C}[X]$ then

$$\frac{(f_2,f_1)_{q,t}}{(1,1)_{q,t}} = \overline{\Big(\frac{(f_1,f_2)_{q,t}}{(1,1)_{q,t}}\Big)}.$$

Proof. (a) Let $f_1, f_2 \in \mathbb{C}[X]$ and $c \in \mathbb{C}[q^{\pm 1}, t^{\pm 1}]$. Then $(cf_1, f_2)_{q,t} = \mathrm{ct}(cf_1\overline{f_2}) = c\cdot\mathrm{ct}(f_1\overline{f_2}) = c(f,g)_{q,t}$ and

$$(f_1, cf_2)_{q,t} = \mathrm{ct}(f_1\overline{cf_2}) = \mathrm{ct}(f_1\overline{c}\overline{f_2}) = \overline{c}\cdot\mathrm{ct}(f_1\overline{f_2}) = \overline{c}(f,g)_{q,t}.$$

(b) Let $f \in \mathbb{C}[X]$ with $f \neq 0$. By clearing denominators appropriately, renormalize f so that f specializes to something nonzero at $q = 1$. If

$$f = \sum_\mu f_\mu x^\mu \qquad \text{then} \qquad (f,f)_{1,1} = (f,f)_{1,1^k} = \sum_\mu |f_\mu|^2 \in \mathbb{R}_{>0}.$$

So $(f,f)_t \neq 0$.

(c) Let $f \in F$ with $f \neq 0$. Since $(f,f)_{q,t} \neq 0$ then there exists $p \in F$ such that $(f,p)_{q,t} \neq 0$. So the restriciton of $(,)_{q,t}$ to F is nondegenerate.

(d) Let

$$f_1 = \sum_\lambda a_\lambda x^\lambda, \qquad f_2 = \sum_\mu b_\mu x^\mu, \qquad \text{and} \qquad \frac{\Delta_{q,t}}{(1,1)_{q,t}} = \frac{\Delta_{q,t}}{\mathrm{ct}(\Delta)} = \sum_{\mu\in\mathbb{Z}^n} d_\mu(q,t)x^\mu.$$

Then

$$\frac{(f_2,f_1)_{q,t}}{(1,1)_{q,t}} = \sum_{\lambda,\mu\in\mathbb{Z}^n} \overline{a_\lambda} b_\mu d_{\lambda-\mu} = \sum_{\lambda,\mu\in\mathbb{Z}^n} \overline{a_\lambda \overline{b_\mu} d_{\mu-\lambda}} = \overline{\Big(\frac{(f_1,f_2)_{q,t}}{(1,1)_{q,t}}\Big)}.$$

□

8.3 Lecture 7: Proof of the inner product characterization of E_μ and P_λ

Proposition 8.3. *Let $\mu \in \mathbb{Z}^n$. The nonsymmetric Macdonald polynomial E_μ is the unique element of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ such that*

(a) $E_\mu = x^\mu + (\text{lower terms})$;

(b) If $\nu \in \mathbb{Z}^n$ and $\nu < \mu$ then $(E_\mu, x^\nu)_{q,t} = 0$.

Proof. Let $W = \mathrm{span}\{x^\mu \mid \nu \in \mathbb{Z}^n \text{ and } \nu \leq \mu\}$,

$$S = \mathrm{span}\{x^\nu \mid \nu \in \mathbb{Z}^n \text{ and } \nu < \mu\} \qquad \text{and} \qquad S^\perp = \{f \in \mathbb{C}[X] \mid \text{if } p \in S \text{ then } (f,p)_{q,t} = 0\}.$$

Since the inner product $(,)_{q,t}$ is nonisotropic then the restriction of $(,)_{q,t}$ to W is nondegenerate and so $\dim(S^\perp) = 1$. Then the normalization of $E_\mu \in S^\perp$ is determined by condition (a). □

Proposition 8.4. *Let $\lambda \in (\mathbb{Z}^n)^+$. The symmetric Macdonald polynomial P_λ is the unique element of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n}$ such that*

(a) $P_\lambda = m_\lambda + (\text{lower terms})$;

(b) If $\gamma \in (\mathbb{Z}^n)^+$ and $\gamma < \lambda$ then $(P_\lambda, m_\gamma)_{q,t} = 0$.

Proof. The proof is completed in the same manner as the proof of Proposition 8.3. □

8.4 Lecture 7: Proof for going up a level from t to qt

Proposition 8.5. *Let $f, g \in \mathbb{C}[X]^{S_n}$ so that f and g are symmetric functions. Then*

$$(f,g)_{q,qt} = \frac{W_0(qt)}{W_0(t^{-1})}(A_\rho f, A_\rho g)_{q,t}.$$

Proof. As in (arhoArhodefn) and (slicksymmA), let

$$c_{w_0}(x;t) = \prod_{1\le i<j\le n} \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}x_ix_j^{-1}}{1-x_ix_j^{-1}} = t^{-\frac{1}{2}\ell(w_0)} \prod_{1\le i<j\le n} \frac{x_j - tx_i}{x_j - x_i} = t^{-\frac{1}{2}\ell(w_0)} \frac{A_\rho(x,t)}{a_\rho}.$$

By (DnabladefnGL),

$$\Delta_{q,t} = \nabla_{q,t} \prod_{i<j} \frac{1-tx_ix_j^{-1}}{1-x_ix_j^{-1}} ? =? t^{\frac{1}{2}\ell(w_0)} \nabla_{q,t} c_{w_0}(x^{-1};t). \qquad \text{(Dnablacomp)}$$

By (DnabladefnGL),

$$\begin{aligned}\nabla_{q,qt} &= \prod_{i\ne j} \frac{(x_ix_j^{-1};q)_\infty}{(qtx_ix_j^{-1};q)_\infty} = \Big(\prod_{i\ne j} \frac{(x_ix_j^{-1};q)_\infty}{(tx_ix_j^{-1};q)_\infty}\Big)\Big(\prod_{i\ne j}(1-tx_ix_j^{-1})\Big) \\ &= \nabla_{q,t}\cdot \prod_{1\le i<j\le n}(1-tx_jx_i^{-1})(1-tx_ix_j^{-1}) = \nabla_{q,t}\cdot \prod_{1\le i<j\le n}(x_j^{-1}-tx_i)(x_j-tx_i)\end{aligned}$$

so that

$$\nabla_{q,qt} = \nabla_{q,t} A_\rho(x;t) A_\rho(x^{-1};t), \qquad \text{(nablaqtreln)}$$

If $f \in \mathbb{C}[X]$ and $w \in A_n$ then

$$\mathrm{ct}(f) = \mathrm{ct}(wf), \qquad \text{and so} \qquad \mathrm{ct}(f) = \frac{1}{n!}\sum_{w\in S_n} \mathrm{ct}(wf). \qquad \text{(cttosymct)}$$

With these identities in hand, let $f, g \in \mathbb{C}[X]^{S_n}$. Then

$$\begin{aligned}
(f,g)_{q,qt} &= \mathrm{ct}(f\bar{g}\Delta_{q,qt}) && \text{(by (innproddefnA))}\\
&= \mathrm{ct}(f\bar{g}\nabla_{q,qt} c_{w_0}(x^{-1};qt)) && \text{(by (Dnablacomp))}\\
&= \mathrm{ct}(f\bar{g}\nabla_{q,t}A_\rho(x;t)A_\rho(x^{-1};t)c_{w_0}(x^{-1};qt)) && \text{(by (nablaqtreln))}\\
&= \frac{1}{|W_0|}\mathrm{ct}\Big(\sum_{w\in W_0} w\big(\, f\bar{g}\nabla_{q,t}A_\rho(x;t)A_\rho(x^{-1};t)c_{w_0}(x^{-1};qt)\ \big)\Big) && \text{(by (cttosymct))}\\
&= \frac{1}{|W_0|}\mathrm{ct}\Big((f\bar{g}\nabla_{q,t}A_\rho(x;t)A_\rho(x^{-1};t)\Big(\sum_{w\in W_0} w(c_{w_0}(x^{-1};qt))\Big)\Big) && (f, g, \nabla_{q,t} \in \mathbb{C}[X]^{W_0})\\
&= \frac{W_0(qt)}{|W_0|}\mathrm{ct}\Big((f\bar{g}\nabla_{q,t}A_\rho(x;t)A_\rho(x^{-1};t)\Big) && \text{(by (Poinbysymm))}
\end{aligned}$$

and

$$\begin{aligned}
(A_\rho f, A_\rho g)_{q,t} &= \mathrm{ct}\big(A_\rho f\ \overline{A_\rho g}\ \Delta_{q,t}\big) && \text{(by (innproddefnA))}\\
&= \mathrm{ct}\big(f\bar{g}\Delta_{q,t}A_\rho(x;t)A_\rho(x^{-1};t^{-1})\big) && \text{(by (invdefns))}\\
&= \mathrm{ct}\Big((f\bar{g}\Delta_{q,t}A_\rho(x;t)A_\rho(x^{-1};t)\frac{A_\rho(x^{-1},t^{-1})}{a_\rho(x^{-1})}\frac{a_\rho(x^{-1})}{A_\rho(x^{-1};t)}\Big)\\
&= \mathrm{ct}\big((f\bar{g}\frac{\Delta_{q,t}}{c_{w_0}(x^{-1};t)}A_\rho(x;t)A_\rho(x^{-1};t)c_{w_0}(x^{-1};t^{-1})\big)\big)\\
&= \mathrm{ct}\big(f\bar{g}\nabla_{q,t}A_\rho(x;t)A_\rho(x^{-1};t)c_{w_0}(x^{-1};t^{-1})\big) && \text{(by (Dnablacomp))}\\
&= \frac{1}{|W_0|}\mathrm{ct}\Big(\sum_{w\in W_0} w\Big(f\bar{g}\nabla_{q,t}A_\rho(x;t)A_\rho(x^{-1};t)c_{w_0}(x^{-1};t^{-1})\Big)\Big) && \text{(by (cttosymct))}\\
&= \frac{1}{|W_0|}\mathrm{ct}\Big((f\bar{g}\nabla_{q,t}A_\rho(x;t)A_\rho(x^{-1};t)\Big(\sum_{w\in W_0} w(c_{w_0}(x^{-1};t^{-1}))\Big)\Big) && (f,g,\nabla_{q,t}\in\mathbb{C}[X]^{W_0})\\
&= \frac{W_0(t^{-1})}{|W_0|}\mathrm{ct}\Big((f\bar{g}\nabla_{q,t}A_\rho(x;t)A_\rho(x^{-1};t)\Big) && \text{(by (Poinbysymm))}
\end{aligned}$$

which gives the result. □

8.5 Lecture 7: Proof of the Weyl character formula for Macdonald polynomials

Theorem 8.6. *Let $\lambda\in\mathbb{Z}^n$ with $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n$. Then*

$$P_\lambda(q,qt)=\frac{A_{\lambda+\rho}(q,t)}{A_\rho(t)}.$$

Proof. Since $A_{\lambda+\rho}=t^{\frac{1}{2}\ell(w_0)}\varepsilon_0E_{\lambda+\rho}$ then $A_{\lambda+\rho}\in\mathbb{C}[X]^{\mathrm{Fer}}$. Thus, by Proposition 7.1,

$$\text{there exists}\qquad f\in\mathbb{C}[X]^{S_n}\qquad\text{such that}\qquad A_{\lambda+\rho}=A_\rho f.$$

If $\mu\in\mathbb{Z}^n$ is such that the coefficient of x^μ in $A_{\lambda+\rho}$ is nonzero then $\mu\le w_0(\lambda+\rho)$. So

$$f=m_\lambda+(\text{lower terms}).$$

The E-expansion for $A_{\lambda+\rho}$ gives that

$$A_{\lambda+\rho}=\sum_{\mu\in S_n(\lambda+\rho)}d^\mu_{\lambda+\rho}E_\mu=E_{w_0(\lambda+\rho)}+(\text{lower terms})$$

and, from the definitions of A_ρ and m_ν,

$$A_\rho m_\nu=x^{w_0(\nu+\rho)}+(\text{lower terms}).$$

Since $(E_{w_0(\lambda+\rho)},x^\gamma)_{q,t}=0$ for $\gamma\in\mathbb{Z}^n$ with $\gamma<w_0(\lambda+\rho)$, then

$$(A_\rho f,A_\rho m_\nu)_{q,t}=(A_{\lambda+\rho},A_\rho m_\nu)_{q,t}=0,\qquad\text{for }\nu\in(\mathbb{Z}^n)^+\text{ with }\nu<\lambda.$$

Thus, by (7.5), since $f\in\mathbb{C}[X]^{S_n}$ and $m_\nu\in\mathbb{C}[X]^{S_n}$ then

$$(f,m_\nu)_{q,t}=(A_\rho f,A_\rho m_\nu)_{q,qt}=0,\qquad\text{for }\nu\in(\mathbb{Z}^n)^+\text{ with }\nu<\lambda.$$

Thus, by Proposition 7.4, $f=P_\lambda(q,t)$. □