13 Lecture 1: Proofs

13.1 Lecture 1: Computation of the inversion set of $t_\mu v$

Proposition 13.1. *Let $\mu \in \mathbb{Z}^n$ and $v \in S_n$. Then*

$$\mathrm{Inv}(t_\mu v) = \Big(\bigcup_{\substack{i<j, v(i)<v(j) \\ \mu_{v(i)} \ge \mu_{v(j)}}} \bigcup_{\ell=0}^{\mu_{v(i)}-\mu_{v(j)}-1} \{(i, j+\ell n)\} \Big) \cup \Big(\bigcup_{\substack{i<j, v(i)>v(j) \\ \mu_{v(i)} \ge \mu_{v(j)}}} \bigcup_{\ell=0}^{\mu_{v(i)}-\mu_{v(j)}} \{(i, j+\ell n)\} \Big)$$
$$\cup \Big(\bigcup_{\substack{i<j, v(i)<v(j) \\ \mu_{v(i)} < \mu_{v(j)}}} \bigcup_{\ell=1}^{\mu_{v(j)}-\mu_{v(i)}} \{((j, i+\ell n)\} \Big) \cup \Big(\bigcup_{\substack{i<j, v(i)>v(j) \\ \mu_{v(i)} < \mu_{v(j)}}} \bigcup_{\ell=1}^{\mu_{v(i)}-\mu_{v(j)}-1} \{((j, i+\ell n)\} \Big)$$

Proof. Let $i, j \in \{1, \ldots, n\}$. If $\ell \in \mathbb{Z}$ then

$$(t_\mu v)(i) = v(i) + \mu_{v(i)} n, \qquad (t_\mu v)(i+\ell n) = v(i) + (\mu_{v(i)} + \ell) n,$$
$$(t_\mu v)(j) = v(j) + \mu_{v(j)} n, \qquad (t_\mu v)(j+\ell n) = v(j) + (\mu_{v(j)} + \ell) n.$$

If $\ell \ge 0$ and $i < j$ then $(t_\mu v)(i) > (t_\mu v)(j+\ell n)$ if

(a) if $v(i) < v(j)$ and $0 \le \ell < \mu_{v(i)} - \mu_{v(j)}$ or (b) if $v(i) > v(j)$ and $0 \le \ell \le \mu_{v(i)} - \mu_{v(j)}$.

If $\ell > 0$ and $j > i$ then $(t_\mu v)(j) > (t_\mu v)(i+\ell n)$

(c) if $v(i) > v(j)$ and $0 < \ell < \mu_{v(j)} - \mu_{v(i)}$ or (d) if $v(i) < v(j)$ and $0 < \ell \le \mu_{v(j)} - \mu_{v(i)}$.

□

13.2 Lecture 1: Computation of the lengths of u_μ and v_μ

Proposition 13.2. *Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\ge 0}$. Let u_μ and v_μ be as defined in (1.7).*

(a) v_μ is the minimal length element of S_n such that $v_\mu \mu$ is (weakly) increasing.

(b) The permutation $v_\mu \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ is given by

$$v_\mu(i) = 1 + \#\{i' \in \{1, \ldots, i-1\} \mid \mu_{i'} \le \mu_i\} + \#\{i' \in \{i+1, \ldots, n\} \mid \mu_{i'} < \mu_i\},$$

(c) The n-periodic permutations $u_\mu \colon \mathbb{Z} \to \mathbb{Z}$ and $u_\mu^{-1} \colon \mathbb{Z} \to \mathbb{Z}$ are given by

$$u_\mu(i) = v_\mu^{-1}(i) + n\mu_i \quad \text{and} \quad u_\mu^{-1}(i) = v_\mu(i) - n\mu_{v_\mu(i)} \quad \text{for } i \in \{1, \ldots, n\}.$$

(d) Let $|\mu_i - \mu_j|$ denote the absolute value of $\mu_i - \mu_j$. Then

$$\ell(t_\mu) = \sum_{\substack{i,j \in \{1,\ldots,n\} \\ i<j}} |\mu_i - \mu_j|, \quad \ell(v_\mu) = \#\{i < j \mid \mu_i > \mu_j\} \quad \text{and} \quad \ell(u_\mu) = \ell(t_\mu) - \ell(v_\mu).$$

Proof. (c) The first formula follows from $u_\mu = t_\mu v_\mu^{-1}$ and (14.6). To verify the second formula:

$$u_\mu^{-1} u_\mu(i) = u_\mu^{-1}(v_\mu^{-1}(i) + n\mu_i) = u_\mu^{-1}(v_\mu^{-1}(i)) + n\mu_i = v_\mu(v_\mu^{-1}(i)) - n\mu_{v_\mu v_\mu^{-1}(i)} + n\mu_i = i.$$

(d) By Proposition 1.1,

$$\mathrm{Inv}(t_\mu) = \Big(\bigcup_{\substack{i<j \\ \mu_i \ge \mu_j}} \bigcup_{\ell=0}^{\mu_j-\mu_i-1} \{(i, j+\ell n)\} \Big) \cup \Big(\bigcup_{\substack{i<j \\ \mu_j<\mu_i}} \bigcup_{\ell=1}^{\mu_i-\mu_j} \{(j, i+\ell n)\} \Big)$$

and so $\ell(t_\mu) = \#\mathrm{Inv}(t_\mu) = \sum_{i<j} |\mu_i - \mu_j|$, which gives the first statement. Since the length of $t_\mu v$ is $\ell(t_\mu v) = \#\mathrm{Inv}(t_\mu v)$ then Proposition 1.1 gives that the minimal length element of the coset $t_\mu S_n$ is the element $t_\mu v_\mu^{-1}$ where, if $i<j$ then $v_\mu^{-1}(i) > v_\mu^{-1}(j)$ if $\mu_{v_\mu^{-1}(i)} < \mu_{v_\mu^{-1}(j)}$ and $v_\mu^{-1}(i) < v_\mu^{-1}(j)$ if $\mu_{v_\mu^{-1}(i)} \ge \mu_{v_\mu^{-1}(j)}$. Thus $v_\mu\mu = v_\mu(\mu_1, \ldots, \mu_n) = (\mu_{v_\mu^{-1}(1)}, \ldots, \mu_{v_\mu^{-1}(n)})$ is in weakly increasing order and $\ell(t_\mu) = \ell(u_\mu) + \ell(v_\mu)$.

(a) now follows from the last line of the proof of (d).

(b) In order for v_μ to rearrange μ into increasing order v_μ must move the ith part of μ to the position just to the right of the number of parts of μ which are less than μ_i, or equal to μ_i and to the left of μ_i. □

13.3 Lecture 1: Proof of the box-greedy reduced word for u_μ

Proposition 13.3. *For a box $(i,j) \in dg(\mu)$ (i.e. $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,\mu_i\}$) define*

$$u_\mu(i,j) = \#\{i' \in \{1,\ldots,i-1\} \mid \mu_{i'} < j \le \mu_i\} + \#\{i' \in \{i+1,\ldots,n\} \mid \mu_{i'} < j-1 < \mu_i\}) \qquad (13.1)$$
$$R_\mu(i,j) = \{\varepsilon^\vee_{v_\mu(i)} - \varepsilon^\vee_1 + (\mu_i - j+1)K, \ldots, \varepsilon^\vee_{v_\mu(i)} - \varepsilon^\vee_{u_\mu(i,j)} + (\mu_i - j+1)K\} \qquad (13.2)$$

(a) The box greedy reduced word for u_μ is

$$u_\mu^\square = \prod_{(i,j)\in dg(\mu)} (s_{u_\mu(i,j)} \cdots s_1 \pi)$$

where the product is over the boxes of μ in increasing cylindrical wrapping order.

(b) The inversion set of u_μ is

$$\mathrm{Inv}(u_\mu) = \bigcup_{(i,j)\in dg(\mu)} R_\mu(i,j) \qquad \text{and} \qquad \ell(u_\mu) = \sum_{(i,j)\in\mu} u_\mu(i,j).$$

Proof. Let $|\mu| = \mu_1 + \cdots + \mu_n$. The proof is by induction on $|\mu|$, the number of boxes of μ.

Let $\mu = (0,\ldots,0,\mu_k,\ldots,\mu_n)$ and let $\nu = \pi^{-1}s_1s_2\cdots s_{k-1}\mu = (0,\ldots,0,\mu_{k+1},\ldots,\mu_n,\mu_k-1)$. From the definition of $u_\mu(i,j)$ in (13.1), $u_\mu(k,1) = k-1$,

$$u_\mu(i,j) = u_\nu(i-1,j) \text{ for } i \in \{k+1,\ldots,n\}, \quad \text{and} \quad u_\mu(k,j) = u_\nu(n,j-1), \text{ if } j \in \{2,\ldots,\mu_k\}.$$

which already establishes (a). Then, using (1.2) gives $v_\mu(i) = i$ for $i \in \{1,\ldots,k-1\}$,

$$v_\mu(i) = v_\nu(i-1) \text{ for } i \in \{k+1,\ldots,n\} \qquad \text{and} \qquad v_\mu(k) = v_\nu(n).$$

These expressions for $u_\mu(i,j)$ and $v_\mu(i)$ in terms of $u_\nu(i,j)$ and $v_\nu(i)$ establish that

$$R_\mu(i,j) = R_\nu(i-1,j), \qquad \text{if } i \ne k, \text{ and}$$
$$R_\mu(k,j) = R_\nu(n,j-1), \qquad \text{if } j \in \{2,\ldots,\mu_k\}.$$

It remains to compute $R_\mu(k,1)$. Since $u_\nu^{-1}\varepsilon_i^\vee = v_\nu t_\nu^{-1}\varepsilon_i^\vee = \varepsilon^\vee_{v_\nu(i)-n\nu_i} = \varepsilon^\vee_{v_\nu(i)} + \nu_i K$ then

$$\begin{aligned}
R_\mu(k,1) &= \{u_\nu^{-1}\pi^{-1}\alpha_1^\vee, \ldots, u_\nu^{-1}\pi^{-1}s_1s_2\cdots s_{k-2}\alpha_{k-1}^\vee\} \\
&= \{u_\nu^{-1}\pi^{-1}(\varepsilon_1^\vee - \varepsilon_2^\vee), \ldots, u_\nu^{-1}\pi^{-1}s_1s_2\cdots s_{k-2}(\varepsilon_{k-1}^\vee - \varepsilon_k^\vee)\} \\
&= \{u_\nu^{-1}((\varepsilon_n^\vee + K) - \varepsilon_1^\vee), \ldots, u_\nu^{-1}((\varepsilon_n^\vee + K) - \varepsilon_{k-1}^\vee)\} \\
&= \{(\varepsilon^\vee_{v_\nu(n)} + \nu_n K + K) - (\varepsilon_1^\vee + \nu_1 K), \ldots, (\varepsilon^\vee_{v_\nu(n)} + \nu_n K + K) - (\varepsilon^\vee_{k-1} + \nu_{k-1}K)\} \\
&= \{\varepsilon^\vee_{v_\mu(k)} - \varepsilon_1^\vee + (\mu_k - 1)K + K, \ldots, \varepsilon^\vee_{v_\mu(k)} - \varepsilon^\vee_{k-1} + (\mu_k - 1)K + K\},
\end{aligned}$$

where the next to last equality uses $\nu_1 = \cdots = \nu_{k-1} = 0$ and $\nu_n = \mu_k - 1$. □