MHS Assignment 1 Question 4 Part A

Let $\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}$.

(a) Let $x \in c_c$ with $x = (x_1, x_2, \ldots)$.

To show: $x \in c_0$.

Since $x \in c_c$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>N}$ then $x_n = 0$.

So, if $\varepsilon \in \mathbb{E}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>N}$ then $|x_n - 0| = |0 - 0| < \varepsilon$.

So $\lim_{n\to\infty} x_n = 0$.

So $x \in c_0$. So $c_c \subseteq c_0$.

Let $x = (1, 10^{-1}, 10^{-2}, 10^{-3}, \ldots)$.

Then $x \in c_0$ since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} 10^{-n} = 0$.

Since there does not exist $N \in \mathbb{Z}_{>0}$ such that $x_N = 0$ then $x \notin c_c$.

So $c_c \neq c_0$

(b) Let $x \in c_c$ with $x = (x_1, x_2, \ldots, x_N, 0, 0, \ldots)$ so that x_N is the last nonzero entry in x.

Then $\|x\|_1 = \sum_{n=1}^{\infty} |x_n| = \sum_{n=1}^{N} |x_N| \in \mathbb{R}_{\geq 0}$.

So $x \in \ell^1$. So $c_c \subseteq \ell^1$.

Let $x = (1, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \ldots)$. Then

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n| = \sum_{n=1}^{\infty} \frac{1}{n^2} \in \mathbb{R}_{\geq 0}.$$

So $x \in \ell^1$.

Since there does not exist $N \in \mathbb{Z}_{>0}$ such that $x_N = 0$ then $x \notin c_c$.

So $\ell^1 \neq c_c$. So $c_c \subsetneq \ell^1$.

(c) To show: $\ell^1 \subseteq \ell^2$.

Assume $x \in \ell^1$ with $x = (x_1, x_2, \ldots)$.

To show: $x \in \ell^2$.

Since $x \in \ell^1$ then $\sum_{n=1}^{\infty} |x_n| \in \mathbb{R}_{\geq 0}$.

So $\lim_{n \to \infty} |x_n| = 0$ and there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>N}$ then $|x_n| < 1$.

Then

$$\sum_{n=1}^{\infty} |x_n|^2 = \sum_{n=1}^{N} |x_n|^2 + \sum_{n=N+1}^{\infty} |x_n|^2$$

$$\leq \sum_{n=1}^{N} |x_n|^2 + \sum_{n=N+1}^{\infty} |x_n|$$

$$\leq \sum_{n=1}^{N} |x_n|^2 + \sum_{n=1}^{\infty} |x_n| \in \mathbb{R}_{\geq 0}.$$

So $\sum_{n=1}^{\infty} |x_n|^2 \in \mathbb{R}_{\geq 0}$ and $x \in \ell^2$.

To show: $\ell^1 \neq \ell^2$.

Let $x = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$.

Then $\|x\|_2^2 = \sum_{n=1}^{\infty} |\frac{1}{n}|^2 \in \mathbb{R}_{\geq 0}$. So $x \in \ell^2$.

Since $\|x\|_1 = \sum_{n=1}^{\infty} \frac{1}{n}$ does not converge $x \notin \ell^1$.

So $\ell^1 \neq \ell^2$.

(d) Let $p \in \mathbb{R}_{>1}$.

To show: $\ell^1 \subseteq \ell^p$

Let $x \in \ell^1$ with $x = (x_1, x_2, \ldots)$.

To show: $x \in \ell^p$.

Since $x \in \ell^1$ then $\sum_{n=1}^{\infty} |x_n| \in \mathbb{R}_{\geq 0}$.

So $\lim_{n \to \infty} |x_n| = 0$ and there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>N}$ then $|x_n| < 1$.

Then

$$\begin{aligned}
\sum_{n=1}^{\infty} |x_n|^p &= \sum_{n=1}^{N} |x_n|^p + \sum_{n=N+1}^{\infty} |x_n|^p \\
&< \sum_{n=1}^{N} |x_n|^p + \sum_{n=N+1}^{\infty} |x_n| \\
&\leq \sum_{n=1}^{N} |x_n|^p + \sum_{n=1}^{\infty} |x_n| \in \mathbb{R}_{\geq 0}
\end{aligned}$$

So $\sum_{n=1}^{\infty} |x_n|^p \in \mathbb{R}_{\geq 0}$ and $x \in \ell^p$

To show: $\ell^1 \neq \ell^p$.

Let $x = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$.

Then $\|x\|_p^p = \sum_{n=1}^{\infty} \left|\frac{1}{n}\right|^p = \sum_{n=1}^{\infty} \frac{1}{n^p} \in \mathbb{R}_{\geq 0}$ since $p > 1$.

So $x \in \ell^p$.

Since $\|x\|_1 = \sum_{n=1}^{\infty} \frac{1}{n}$ does not converge then $x \notin \ell^1$.

So $\ell^1 \neq \ell^p$.

(f) To show: $\ell^q \subseteq c_0$, where $q \in \mathbb{R}_{>1}$.

Let $x \in \ell^q$ with $x = (x_1, x_2, \ldots)$.

To show: $x \in c_0$.

To show: $\lim_{n\to\infty} x_n = 0$.

Since $x \in \ell^q$ then $\|x\|_q^q = \sum_{n=1}^{\infty} |x_n|^q \in \mathbb{R}_{\geq 0}$.

So $\lim_{n\to\infty} |x_n|^q = 0$.

By Theorem 17.19 of the notes on Real numbers on the course website the function

$$\begin{aligned} \mathbb{R}_{\geq 0} &\to \mathbb{R}_{\geq 0} \\ x &\mapsto x^q \end{aligned}$$ satisfies if $x < y$ then $x^q < y^q$ and is bijective.

So $\lim_{n\to\infty} |x_n| = 0$.

So $x \in c_0$.

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To show: $\ell^2 \neq c_0$

Let $x = \left(1, \frac{1}{2^{1/2}}, \frac{1}{3^{1/2}}, \frac{1}{4^{1/2}}, \ldots\right)$.

Then $\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} \frac{1}{n^{1/2}} = 0$. So $x \in c_0$.

Since $\|x\|_2^2 = \sum_{n=1}^{\infty} \left|\frac{1}{n^{1/2}}\right|^2 = \sum_{n=1}^{\infty} \frac{1}{n}$ does not converge

then $x \notin \ell^2$.

So $\ell^2 \subseteq c_0$ and $\ell^2 \neq c_0$.

(g) To show: $c_0 \subseteq \ell^\infty$

Let $x \in c_0$ with $x = (x_1, x_2, \ldots)$.

Then $\lim_{n\to\infty} x_n = 0$.

So there exists $N \in \mathbb{Z}_{>0}$ such that $|x_N| < 1$.

Then

$$\|x\|_\infty = \sup\{|x_1|, |x_2|, \ldots\}$$
$$\leq \sup\{|x_1|, |x_2|, \ldots, |x_N|, 1\} \in \mathbb{R}_{\geq 0}$$

So $x \in \ell^\infty$.

So $c_0 \subseteq \ell^\infty$.

Let $x = (1, 1, 1, \ldots)$. Then $\|x\|_\infty = \sup\{1\} = 1 \in \mathbb{R}_{\geq 0}$.

So $x \in \ell^\infty$.

Since $\lim_{n\to\infty} x_n = 1$ then $x \notin c_0$

So $\ell^\infty \neq c_0$.