MAST30026 Metric and Hilbert Spaces

Assignment 2: Updated 26.08.2022

Due: 4pm Thursday September 8, 2022

As usual, part of the exercise is to correct any typos and, in your solutions, to explain carefully to the marker what you corrected and why.

Question 1. (Parseval's identities) Let $S = \{e_1, e_2, \dots\}$ be a countable orthonormal basis for a separable Hilbert space H. Prove that

(a) $x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ (Fourier expansion),

(b) $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$ (Parseval's identity for norms),

(c) $\langle x, y \rangle = \sum_{n=1}^{\infty} \langle x, e_n \rangle \overline{\langle y, e_n \rangle}$ (Parseval's identity for inner products).

Question 2 (Fourier decomposition for complex valued funcrions). Let

$$L^2([0, 2\pi]) = \{f\colon [0, 2\pi] \to \mathbb{C} \mid \ \|f\|_2 \text{ exists in } \mathbb{R}_{\ge 0}\}$$

where the norm $\| \ \|_2\colon L^2([0, 2\pi]) \to \mathbb{R}_{\ge 0}$ and the sesquilinear form $\langle , \rangle\colon L^2([0, 2\pi]) \times L^2([0, 2\pi]) \to \mathbb{C}$ are given by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)} dt \qquad \text{and} \qquad \|f\|_2 = \sqrt{\langle f, f \rangle}.$$

Define

$$e_m(t) = e^{imt}, \qquad \text{for } m \in \mathbb{Z} \text{ and } t \in \mathbb{R}_{[0,2\pi]}.$$

Let $L\colon L^2(\mathbb{R}_{[0,2\pi]}) \to L^2(\mathbb{R}_{[0,2\pi]})$ be the operator on $L^2(\mathbb{R}_{[0,2\pi]})$ given by $\quad L = \frac{d}{dt}$.

(a) Let $m \in \mathbb{Z}$. Show that e_m is an eigenvector of L and compute the eigenvalue.

(b) Show that $(e_0, e_1, e_{-1}, e_2, e_{-2}, \dots)$ is an orthonormal sequence in $L^2([0, 2\pi])$.

(c) Determine $a_0, a_1, a_{-1}, a_2, a_{-2}, \dots \in \mathbb{C}$ such that $t = a_0 e_0 + a_1 e_1 + a_{-1} e_{-1} + a_2 e_2 + a_{-2} e_{-2} + \cdots$.

(d) Determine $c_0, c_1, c_{-1}, c_2, c_{-2}, \dots \in \mathbb{C}$ such that $t^2 = c_0 e_0 + c_1 e_1 + c_{-1} e_{-1} + c_2 e_2 + c_{-2} e_{-2} + \cdots$.

Question 3 (Fourier decomposition for real valued functions). Let

$$L^2([0, 2\pi]) = \{f\colon [0, 2\pi] \to \mathbb{R} \mid \ \|f\|_2 \text{ exists in } \mathbb{R}_{\ge 0}\}$$

where the norm $\| \ \|_2\colon L^2([0, 2\pi]) \to \mathbb{R}_{\ge 0}$ and the sesquilinear form $\langle , \rangle\colon L^2([0, 2\pi]) \times L^2([0, 2\pi]) \to \mathbb{R}$ are given by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x)g(x) dx \qquad \text{and} \qquad \|f\|_2 = \sqrt{\langle f, f \rangle}.$$

Let $s_0(t) = 1$ for $t \in \mathbb{R}_{[0,2\pi]}$. For $n \in \mathbb{Z}_{>0}$ and $t \in \mathbb{R}_{[0,2\pi]}$ define

$$s_n(t) = \sqrt{2}\sin(nt), \qquad \text{and} \qquad s_{-n}(t) = \sqrt{2}\cos(nt).$$

(a) Show that $(s_0, s_1, s_{-1}, s_2, s_{-2}, \ldots)$ form an orthonormal sequence in $L^2([0, 2\pi])$.

(b) Let $L\colon L^2(\mathbb{R}_{[0,2\pi]}) \to L^2(\mathbb{R}_{[0,2\pi]})$ be the operator on $L^2(\mathbb{R}_{[0,2\pi]})$ given by

$$L = \frac{d^2}{dt^2}.$$

Let $n \in \mathbb{Z}_{\geq 0}$. Show that s_n and s_{-n} are eigenvectors of L and compute the eigenvalues.

(c) Carefully graph the function $f\colon \mathbb{R}_{[0,2\pi]} \to \mathbb{R}$ given by $f(t) = 2\pi t - t^2$.

(d) Expand the function $f\colon \mathbb{R}_{[0,2\pi]} \to \mathbb{R}$ given by $f(t) = 2\pi t - t^2$ in terms of the orthonormal sequence $(s_0, s_1, s_{-1}, s_2, s_{-2}, \ldots)$.

(e) Carefully compute $A, B \in \mathbb{C}$ such that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = A \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = B.$$

Question 4 (Hermite polynomials). Define a norm $\| \ \|_w\colon \mathbb{C}[x] \to \mathbb{R}_{\geq 0}$ and a sesquilinear form $\langle , \rangle\colon \mathbb{C}[x] \times \mathbb{C}[x] \to \mathbb{C}$ by

$$\langle f, g\rangle_w = \int_{-\infty}^{\infty} f(x)\overline{g(x)} e^{-\frac{1}{2}x^2} dx \qquad \text{and} \qquad \|f\|_w = \sqrt{\langle f, f\rangle_w}.$$

Define the Hermite polynomials $P_n(x)$ by

$$P_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}\big(e^{-\frac{1}{2}x^2}\big), \qquad \text{for } n \in \mathbb{Z}_{>0}.$$

Let $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}_{>0}$.

(a) Carefully graph the "bell curve" $w\colon \mathbb{R} \to \mathbb{R}$ and the "normal distribution" $N_{\mu,\sigma}\colon \mathbb{R} \to \mathbb{R}$ which are given by

$$w(x) = e^{-\frac{1}{2}x^2} \qquad \text{and} \qquad N_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

(b) Carefully prove that

$$\langle 1, 1\rangle_w = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi} \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1.$$

Explain why these integrals are vital to every data analyst, statistician and probabilist.

(c) Define operators $D\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$, $X\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$, $S\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ and $E\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ by

$$Df = \frac{df}{dx}, \qquad Xf = xf, \qquad Sf = e^{\frac{1}{2}x^2} f, \qquad \text{and} \qquad E = SDS^{-1}.$$

Carefully prove the following identities:

$$E^n = SD^nS^{-1}, \qquad XD = DX - 1, \qquad SD = DS - XS, \qquad SX = XS, \qquad SDS^{-1} = D - X,$$

$$DE^n = E^{n+1} + XE^n, \qquad XD^n = D^nX - nD^{n-1},$$
$$DX^n = X^nD + nX^{n-1}, \qquad XE^n = E^nX - nE^{n-1}.$$

(c) Carefully compute P_0, P_1, P_2, P_3 and P_4 and prove that

$$\frac{d}{dx}P_n(x) = xP_n(x) - P_{n+1}(x), \qquad P_{n+1}(x) = xP_n(x) - nP_{n-1}(x), \quad \text{and} \qquad \frac{d}{dx}P_n(x) = nP_{n-1}(x).$$

(d) Show that if $k, n \in \mathbb{Z}_{\geq 0}$ and $k < n$ then $\langle P_k, P_n\rangle_w = 0$.

(e) Prove carefully that if $n \in \mathbb{Z}_{\geq 0}$ then $\langle P_n, P_n\rangle_w = \sqrt{2\pi}\, n!$.

(f) Prove that $(P_0, P_1, P_2, \ldots)$ is an orthogonal basis of $\mathbb{C}[x]$ with respect to $\langle,\rangle_w$.

Question 5 (The quantum harmonic oscillator). Let

$$L^2(\mathbb{R}) = \{f\colon \mathbb{R} \to \mathbb{C} \mid \ \|f\| \text{ exists in } \mathbb{R}_{\geq 0}\}$$

where the norm $\|\ \|\colon L^2(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ and the sesquilinear form $\langle,\rangle\colon L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to \mathbb{C}$ are given by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx \qquad \text{and} \qquad \|f\| = \sqrt{\langle f, f\rangle}.$$

Let $m, \omega, \hbar \in \mathbb{R}_{>0}$ and define

$$h_r(x) = \frac{1}{\sqrt{r!}}\Big(\frac{m\omega}{\pi\hbar}\Big)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} P_r\Big(\big(\frac{2m\omega}{\hbar}\big)^{\frac{1}{2}}x\Big), \qquad \text{for } r \in \mathbb{Z}_{\geq 0},$$

where $P_r(x)$ are the Hermite polynomials of Question 4. Let $i = \sqrt{-1}$ and define operators

$$p = -i\hbar\frac{d}{dx} \qquad \text{and} \qquad H = \frac{1}{2m}p^2 + \tfrac{1}{2}m\omega^2x^2.$$

More explicitly, if $f \in L^2(\mathbb{R})$ then $\qquad Hf = \dfrac{-\hbar^2}{2m}\dfrac{d^2f}{dx^2} + \tfrac{1}{2}m\omega^2x^2f.$

(a) Show that $(h_0, h_1, h_2, \ldots)$ is an orthonormal sequence in $L^2(\mathbb{R})$.

(b) Define operators a, $a^\dagger$ and N on $L^2(\mathbb{R})$ by

$$a = \Big(\frac{m\omega}{2\hbar}\Big)^{\frac{1}{2}}\big(x + i\frac{1}{m\omega}p\big), \qquad a^\dagger = \Big(\frac{m\omega}{2\hbar}\Big)^{\frac{1}{2}}\big(x - i\frac{1}{m\omega}p\big), \qquad \text{and} \qquad N = a^\dagger a.$$

Prove that

$$N = \Big(\frac{m\omega}{2\hbar}\Big)\Big(x^2 - \frac{\hbar^2}{m^2\omega^2}\frac{d^2}{dx^2}\Big) - \tfrac{1}{2}, \qquad aa^\dagger - a^\dagger a = 1, \qquad Na^\dagger - a^\dagger N = a^\dagger, \qquad Na - aN = -a.$$

(c) Show that, as operators on $L^2(\mathbb{R})$,

$$H = \hbar\omega\big(N + \tfrac{1}{2}\big).$$

(d) Prove carefully that

$$a^\dagger h_n = (n+1)^{\frac{1}{2}}h_{n+1}, \qquad ah_n = n^{\frac{1}{2}}h_{n-1}, \qquad \text{and} \qquad Nh_n = nh_n.$$

(e) Show that h_n is an eigenvector of H and compute the eigenvalue.

2.3 Assignment 3