MAST30026 Metric and Hilbert Spaces

Assignment 3

Due: 4pm Thursday October 6, 2022

Question 1. The *Zariski topology* (also called the *cofinite topology*) on $\mathbb{R}$ is

$$\mathcal{T} = \{U \subseteq \mathbb{R} \mid U^c \text{ is finite}\}, \qquad \text{where } U^c \text{ denotes the complement of } U \text{ in } \mathbb{R}.$$

(a) Prove carefully that $\mathcal{T}$ is a topology on $\mathbb{R}$.

(b) Prove carefully that $(\mathbb{R}, \mathcal{T})$ is not Hausdorff.

(c) Determine (with proof) $\overline{\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}}$ in $(\mathbb{R}, \mathcal{T})$.

(d) Determine (with proof) the connected sets in $(\mathbb{R}, \mathcal{T})$.

(e) Determine (with proof) the compact sets in $(\mathbb{R}, \mathcal{T})$.

(f) Find (with proof) a metric $d\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}_{\geq 0}$ such that the metric space topology on $\mathbb{R}$ determined by the metric d is the same as $\mathcal{T}$.

Question 2. Let $X = \{0_1\} \cup \{0_2\} \cup \mathbb{R}_{>0}$ and define a function $d\colon X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ by

$$d(y,x) = d(x,y) \qquad \text{and} \qquad d(x,y) = \begin{cases} |y-x|, & \text{if } x, y \in \mathbb{R}_{>0}, \\ y, & \text{if } y \in \mathbb{R}_{>0} \text{ and } x \in \{0_1, 0_2\}, \\ 0, & \text{if } x = 0_1 \text{ and } y = 0_1, \\ 0, & \text{if } x = 0_2 \text{ and } y = 0_2, \\ \infty, & \text{if } x = 0_1 \text{ and } y = 0_2. \end{cases}$$

Let $\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}$ and let $B_\epsilon(x) = \{y \in X \mid d(y,x) < \epsilon\}$ for $\epsilon \in \mathbb{E}$ and $x \in X$. Let $\mathcal{T}$ be the topology on X generated by

$$\mathcal{B} = \{B_\epsilon(x) \mid \epsilon \in \mathbb{E}, x \in X\}.$$

(a) Prove carefully that $(X, \mathcal{T})$ is not Hausdorff.

(b) Determine (with proof) $\overline{\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}}$ in $(X, \mathcal{T})$.

Question 3 Let $(X, \mathcal{T})$ be a topological space. Let $S \subseteq X$.

(a) Carefully define what it means for S to be connected.

(b) Carefully define what it means for S to be path connected.

(c) Prove carefully that if S is path connected then S is connected.

(d) Give an explicit example of a topological space $(X, \mathcal{T})$ and $S \subseteq X$ such that S is connected but S is not path connected. Be sure to prove carefully that the S in your example is connected and is not path connected.