14 Function spaces

14.1 $\mathbb{R}^2$

The favourite example is $\mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$ with *addition* and *scalar multiplication* given by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \qquad \text{and} \qquad c(x_1, x_2) = (cx_1, cx_2), \quad \text{for } c \in \mathbb{R},$$

with *inner product*

$$\begin{matrix} \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}_{\geq 0} \\ (x, y) & \longmapsto & \langle x, y \rangle \end{matrix} \qquad \text{given by} \qquad \langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 + x_2 y_2,$$

with *norm*

$$\begin{matrix} \mathbb{R}^2 & \longrightarrow & \mathbb{R}_{\geq 0} \\ x & \longmapsto & \|x\| \end{matrix} \qquad \text{given by} \qquad \|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2},$$

with *metric* $d \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ given by

$$d((x_1, x_2), (y_1, y_2)) = \|(x_1, x_2) - (y_1, y_2)\| = \|(x_1 - y_1, x_2 - y_2)\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

with *uniformity* given by

$$\mathcal{X} = \{\text{subsets of } \mathbb{R}^2 \times \mathbb{R}^2 \text{ which contain a set } B_\epsilon\} \qquad \text{where}$$

$$B_\epsilon = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid d(x, y) < \epsilon\} \text{ for } \epsilon \in \mathbb{R}_{>0},$$

and with *topology* given by

$$\mathcal{T} = \{\text{unions of open balls}\},$$

where the *set of open balls* is

$$\mathcal{B} = \{B_\epsilon(x) \mid \epsilon \in \mathbb{R}_{>0}, x \in \mathbb{R}^2\} \qquad \text{and} \quad B_\epsilon(x) = \{y \in \mathbb{R}^2 \mid d(y, x) < \epsilon\}$$

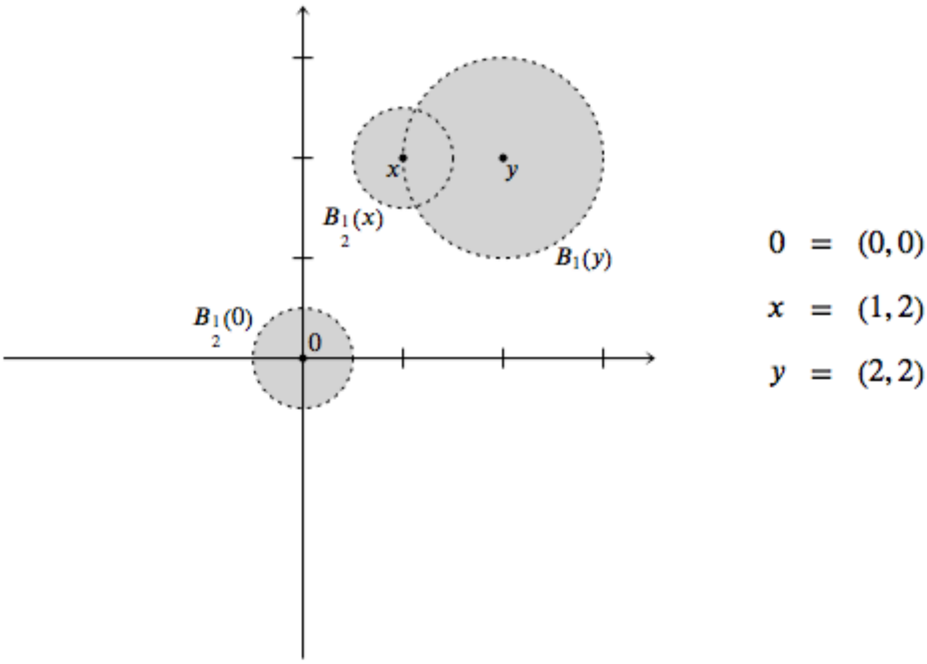
is the *ball of radius ϵ centered at x* (yes, to stress, strongly, that we normally assume that the set $\mathbb{R}^2$ is endowed with lots of extra structures this is, intentionally, a very run-on sentence).

14.2 Favourite vector spaces $\mathbb{R}^n$ and $\mathbb{R}^\infty$, the norms $\| \ \|_p$ and the spaces ℓ^p

14.2.1 Sequences are functions

Let $n \in \mathbb{Z}_{>0}$. Identify n-tuples $(x_1, \ldots, x_n)$ of elements of $\mathbb{R}$ with functions $\vec{x} \colon \{1, 2, \ldots, n\} \to \mathbb{R}$ so that

$$\text{the } n\text{-tuple } (x_1, \ldots, x_n) \qquad \text{is identified with the function} \quad \begin{matrix} \vec{x}\colon & \{1, \ldots, n\} & \to & \mathbb{R} \\ & i & \mapsto & x_i \end{matrix}$$



Open balls in $\mathbb{R}^2$.

14.2.2 The vector space $\mathbb{R}^n$

Let $n \in \mathbb{Z}_{\geq 0}$. The space of functions from $\{1, 2, \ldots, n\}$ to $\mathbb{R}$ is

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\} = \{\text{functions } \vec{x}\colon \{1, \ldots, n\} \to \mathbb{R}\}.$$

with *addition and scalar multiplication* given by

$$\begin{array}{c}(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \quad \text{and} \\ c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, x_n), \qquad \text{for } c \in \mathbb{R},\end{array}$$

and with *inner product* $\langle\, , \rangle\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by

$$\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \rangle = x_1 y_1 + x_2 y_2 + \cdots x_n y_n,$$

with *norm*

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}_{\geq 0} \\ x & \longmapsto & \|x\| \end{array} \qquad \text{given by} \qquad \|(x_1, x_2, \ldots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

with *metric* $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by

$$\begin{aligned} d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) &= \|(x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n)\| \\ &= \|(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}, \end{aligned}$$

with *uniformity* given by

$$\mathcal{X} = \{\text{subsets of } \mathbb{R}^n \times \mathbb{R}^n \text{ which contain a set } B_\epsilon\} \qquad \text{where}$$

$$B_\epsilon = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mid d(x,y) < \epsilon\} \text{ for } \epsilon \in \mathbb{R}_{>0},$$

and with *topology* given by

$$\mathcal{T} = \{\text{unions of open balls}\},$$

where the *set of open balls* is

$$\mathcal{B} = \{B_\epsilon(x) \mid \epsilon \in \mathbb{R}_{>0}, x \in \mathbb{R}^n\} \qquad \text{and} \qquad B_\epsilon(x) = \{y \in \mathbb{R}^n \mid d(y,x) < \epsilon\}$$

is the *ball of radius* ϵ *centered at* x (yes, to stress, strongly, that we normally assume that the set $\mathbb{R}^n$ is endowed with lots of extra structures this is, intentionally, a very run-on sentence).

14.2.3 The vector space $\mathbb{R}^\infty$

Let

$$\mathbb{R}^\infty = \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{R}\} = \{\vec{x} \colon \mathbb{Z}_{>0} \to \mathbb{R}\}$$

with addition and scalar multiplication given by

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (x_1 + y_1, x_2 + y_2, \ldots) \quad \text{and} \quad c(x_1, x_2, \ldots) = (cx_1, cx_2, \ldots),$$

for $(x_1, x_2, \ldots), (y_1, y_2, \ldots) \in \mathbb{R}^\infty$ and $c \in \mathbb{R}$.

The vector space of *sequences with finite support* is

$$c_c = \{(x_1, x_2, \ldots) \in \mathbb{R}^\infty \mid \text{all but a finite number of the } x_i \text{ are } 0\},$$

the space of sequences that are eventually 0. The vector space

$$c_0 = \Big\{(x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \lim_{n\to\infty} x_n = 0\Big\} \qquad \text{has } c_c \subsetneq c_0 \subsetneq \mathbb{R}^\infty.$$

14.2.4 The spaces ℓ^p

Let $p \in \mathbb{R}_{\geq 1}$ and define

$$\ell^p = \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \|\vec{x}\|_p < \infty\}, \quad \text{where} \qquad \|\vec{x}\|_p = \Bigg(\sum_{i\in\mathbb{Z}_{>0}} |x_i|^p\Bigg)^{1/p}$$

for a sequence $\vec{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\infty$.

14.2.5 The spaces ℓ^∞, c_0 and ℓ^1

Define

$$\ell^\infty = \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \|\vec{x}\|_\infty < \infty\}, \quad \text{where} \qquad \|\vec{x}\|_\infty = \sup\{|x_1|, |x_2|, \ldots\},$$

for a sequence $\vec{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\infty$. Define

$$c_0 = \Big\{(x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \lim_{n\to\infty} x_n = 0\Big\}, \quad \text{with norm} \quad \|\vec{x}\|_\infty = \sup\{|x_1|, |x_2|, \ldots\},$$

Note that

$$\ell^1 = \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \|\vec{x}\|_1 < \infty\}, \quad \text{where} \qquad \|\vec{x}\|_1 = |x_1| + |x_2| + \cdots,$$

for a sequence $\vec{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\infty$.

14.2.6 The space ℓ^2

Define

$$\ell^2 = \{\vec{x} = (x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \|\vec{x}\|_2 < \infty\}, \quad \text{where} \qquad \|\vec{x}\|_2 = \left(\sum_{i\in\mathbb{Z}_{>0}} |x_i|^2\right)^{1/2},$$

and define an inner product on ℓ^2,

$$\langle\, , \rangle\colon \ell^2 \times \ell^2 \to \mathbb{R}_{\geq 0} \qquad \text{given by} \qquad \langle \vec{x}, \vec{y}\rangle = \sum_{i\in\mathbb{Z}_{>0}} x_i y_i,$$

for $\vec{x} = (x_1, x_2, \ldots)$ and $\vec{y} = (y_1, y_2, \ldots)$ in ℓ^2.

14.3 Some proofs

14.3.1 The Hölder and Minkowski inequalities

Theorem 14.1. *Let $q \in \mathbb{R}_{\geq 1}$ and let $p \in \mathbb{R}_{>1} \cup \{\infty\}$ be given by $\frac{1}{p} + \frac{1}{q} = 1$. Let*

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \qquad \text{for } x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.,$$

and define

$$\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n, \qquad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{R}^n \text{ and } y = (y_1, \ldots, y_n) \in \mathbb{R}^n.$$

(a) Let $a, b \in \mathbb{R}_{>0}$. Then

$$a^{\frac{1}{p}}\, b^{\frac{1}{q}} \leq \tfrac{1}{p}\, a + \tfrac{1}{q}\, b.$$

(b) Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Then

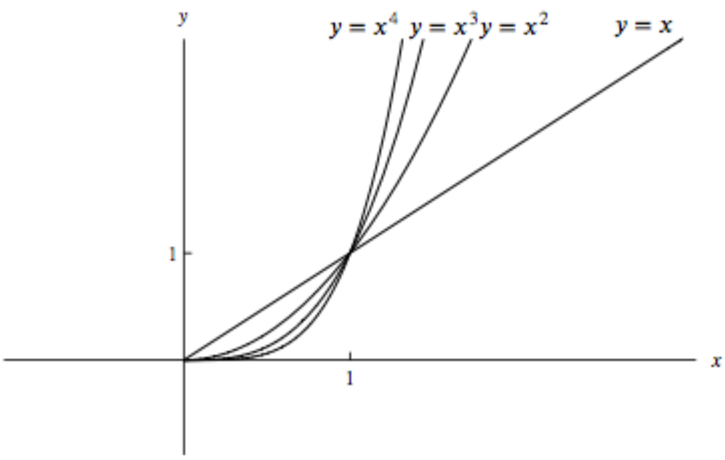
$$|\langle x, y\rangle| \leq \|x\|_p\|y\|_q \qquad \text{and} \qquad \|x+y\|_p \leq \|x\|_p + \|y\|_p.$$

(b′) Let $x = (x_1, x_2, \ldots) \in \ell^p$, $y = (y_1, y_2, \ldots) \in \ell^q$ and $\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots$. Then

$$|\langle x, y\rangle| \leq \|x\|_p\|y\|_q \qquad \text{and} \qquad \|x+y\|_p \leq \|x\|_p + \|y\|_p.$$

Proof. (a) The functions $h\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by

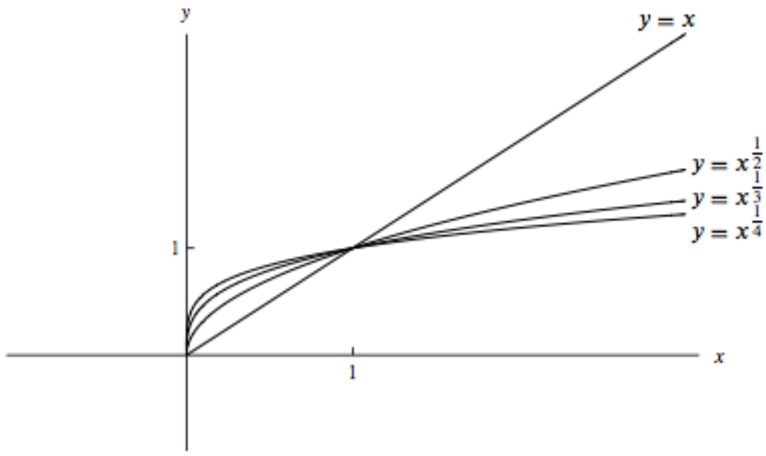
$h(x) = x^q$



are increasing with $h(1) = 1$.

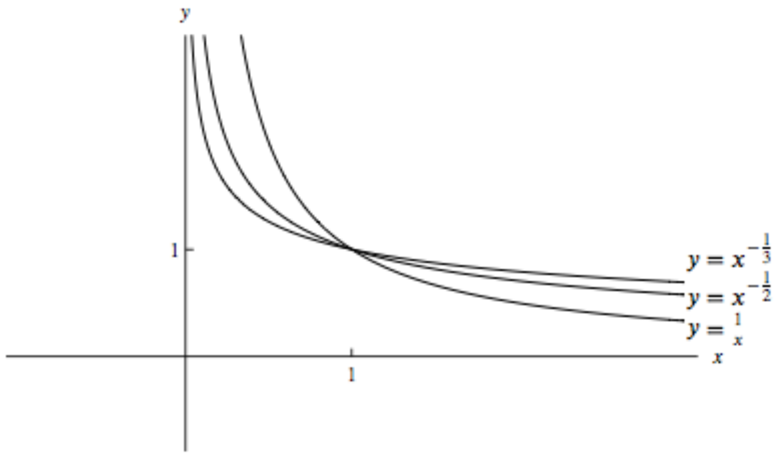
The functions $h\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ given by

$h(x) = x^{\frac{1}{q}}$



are increasing with $h(1) = 1$.
The functions $h\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ given by

$h(x) = x^{-\frac{1}{q}}$



are decreasing with $h(1) = 1$.
Thus the functions $g\colon \mathbb{R}_{\ge 0} \to \mathbb{R}$ given by $g(x) = x^{-1/q} - 1$ are decreasing with $g(1) = 0$.
So

$$x^{-\frac{1}{q}} - 1 \le 0, \qquad \text{for } x \in \mathbb{R}_{\ge 1}.$$

If $f\colon \mathbb{R}_{>0} \to \mathbb{R}$ is given by

$$f(x) = x^{1/p} - \tfrac{1}{p}x, \qquad \text{then} \quad \frac{df}{dx} = \tfrac{1}{p}x^{\frac{1}{p}-1} - \tfrac{1}{p} = \tfrac{1}{p}\big(x^{-\frac{1}{q}} - 1\big),$$

and so f is decreasing for $x \in \mathbb{R}_{>1}$ and $f(1) = 1 - \frac{1}{p} = \frac{1}{q}$.
So

$$x^{1/p} - \tfrac{1}{p}x \le \tfrac{1}{q}, \quad \text{for } x \in \mathbb{R}_{\ge 1}.$$

Let $a, b \in \mathbb{R}_{>0}$ with $a \geq b$ and let $x = \dfrac{a}{b}$. Then

$$\tfrac{1}{q} \geq \Big(\frac{a}{b}\Big)^{\frac{1}{p}} - \tfrac{1}{p}\Big(\frac{a}{b}\Big) = \frac{1}{b}\Big(a^{\frac{1}{p}}b^{-\frac{1}{p}+1} - \tfrac{1}{p}a\Big) = \frac{1}{b}\Big(a^{\frac{1}{p}}b^{\frac{1}{q}} - \tfrac{1}{p}a\Big).$$

So

$$\tfrac{1}{p}\,a + \tfrac{1}{q}\,b \geq a^{\frac{1}{p}}\,b^{\frac{1}{q}}, \quad \text{for } a, b \in \mathbb{R}_{>0}.$$

(b) Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. By part (a),

$$\frac{|x_i y_i|}{\|x\|_p\,\|y\|_q} \leq \tfrac{1}{p}\left(\frac{|x_i|}{\|x\|_p}\right)^p + \tfrac{1}{q}\left(\frac{|y_i|}{\|y\|_q}\right)^q.$$

So

$$\sum_{i=1}^n \frac{|x_i y_i|}{\|x\|_p\,\|y\|_q} \leq \sum_{i=1}^n \tfrac{1}{p}\left(\frac{|x_i|}{\|x\|_p}\right)^p + \tfrac{1}{q}\left(\frac{|y_i|}{\|y\|_q}\right)^q = \tfrac{1}{p} + \tfrac{1}{q} = 1.$$

So

$$\sum_{i=1}^n |x_i y_i| \leq \|x\|_p\,\|y\|_q.$$

So

$$\Big|\sum_{i=1}^n x_i y_i\Big| \leq \sum_{i=1}^n |x_i y_i| \leq \|x\|_p\,\|y\|_q.$$

(c) Using $|x_i + y_i| \leq |x_i| + |y_i|$ and $p - 1 = p(1 - \frac{1}{p}) = p\,\frac{1}{q} = \frac{p}{q}$, and

$$\begin{aligned}\Big\|\big(|x_1+y_1|^{\frac{1}{q}}, \ldots, |x_n+y_n|^{\frac{p}{q}}\big)\Big\|_q &= \left(\sum_{i=1}^n \big(|x_i+y_i|^{\frac{p}{q}}\big)^q\right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^n |x_i+y_i|^p\right)^{\frac{1}{p}\cdot\frac{p}{q}} = \big(\|x+y\|_p\big)^{\frac{p}{q}},\end{aligned}$$

and the identity from part (b), gives

$$\begin{aligned}\|x+y\|_p^p &= \sum_{i=1}^n |x_i+y_i|^p = \sum_{i=1}^n |x_i+y_i|\,|x_i+y_i|^{p-1} \leq \sum_{i=1}^n (|x_i|+|y_i|)|x_i+y_i|^{\frac{p}{q}} \\ &= \sum_{i=1}^n |x_i|\,|x_i+y_i|^{\frac{p}{q}} + \sum_{i=1}^n |y_i|\,|x_i+y_i|^{\frac{p}{q}} \\ &\leq \|x\|_p\|\big(|x_1+y_1|^{\frac{p}{q}}, \ldots, |x_n+y_n|^{\frac{p}{q}}\big)\|_q + \|y\|_p\|\big(|x_1+y_1|^{\frac{p}{q}}, \ldots, |x_n+y_n|^{\frac{p}{q}}\big)\|_q \\ &= \|x\|_p\big(\|x+y\|_p\big)^{\frac{p}{q}} + \|y\|_p\big(\|x+y\|_p\big)^{\frac{p}{q}} = (\|x\|_p + \|y\|_q)\|x+y\|_p^{p-1}.\end{aligned}$$

Dividing both sides by $|x+y|_p^{p-1}$, then

$$\|x+y\|_p \leq \|x\|_p + \|q\|_p.$$

PUT IN THE PROOFS OF (b′) AND (c′). ☐

14.3.2 The dual of ℓ^p

Theorem 14.2. *Let $p \in \mathbb{R}_{>1}$ and let $q \in \mathbb{R}_{>1}$ be defined by*

$$\frac{1}{p} + \frac{1}{q} = 1. \qquad \textit{Then} \quad (\ell^p)^* = \ell^q.$$

Proof.

To show: ℓ^q is the dual of ℓ^p.
To show: $\ell^q = B(\ell^p, \mathbb{R})$.
Define

$$\begin{array}{rccl} \varphi\colon & \ell^q & \longrightarrow & B(\ell^p, \mathbb{R}) \\ & y & \longmapsto & \begin{array}{rccc} \varphi_y\colon & \ell^p & \to & \mathbb{R} \\ & x & \mapsto & \langle y, x\rangle \end{array} \end{array}$$

where

$$\langle y, x\rangle = \sum_{i\in\mathbb{Z}_{>0}} y_i x_i, \qquad \text{if } y = (y_1, y_2, \ldots) \text{ and } x = (x_1, x_2, \ldots).$$

To show: (a) φ is a linear transformation.
(b) φ is invertible.
(c) If $y \in \ell^q$ then $\|\varphi_y\| = \|y\|$.

(a) To show: (aa) If $y_1, y_2 \in \ell^q$ then $\varphi(y_1 + y_2) = \varphi(y_1) + \varphi(y_2)$.
(ab) If $y \in \ell^q$ and $c \in \mathbb{R}$ then $\varphi(cy) = c\varphi(y)$.

(aa) Assume $y_1, y_2, \in \ell^q$.
To show: $\varphi(y_1 + y_2) = \varphi(y_1) + \varphi(y_2)$.
To show: If $x \in \ell^p$ then $\varphi(y_1 + y_2) = \varphi(y_1) + \varphi(y_2)$.
Assume $x \in \ell^p$.
To show: $\varphi_{y_1+y_2}(x) = (\varphi_{y_1} + \varphi_{y_2})(x)$.

$$\varphi_{y_1+y_2}(x) = \langle y_1 + y_2, x\rangle = \langle y_1, x\rangle + \langle y_2, x\rangle = \varphi_{y_1}(x) + \varphi_{y_2}(x) = (\varphi_{y_1} + \varphi_{y_2})(x).$$

(ab) Assume $y \in \ell^q$ and $c \in \mathbb{R}$.
To show: $\varphi(cy) = c\varphi(y)$.
To show: If $x \in \ell^p$ then $\varphi_{cy}(x) = (c\varphi_y)(x)$.
Assume $x \in \ell^p$.
To show $\varphi_{cy}(x) = (c\varphi_y)(x)$.

$$\varphi_{cy}(x) = \langle cy, x\rangle = c\langle y, x\rangle = c(\varphi_y(x)) = (c\varphi_y)(x).$$

So φ is a linear transformation.

(b) To show: $\varphi\colon \ell^q \to B(\ell^p, \mathbb{R})$ is invertible.
To show: There exists $\psi\colon B(\ell^p, \mathbb{R}) \to \ell^q$ such that $\varphi \circ \psi = \text{id}$ and $\psi \circ \varphi = \text{id}$.
Let $\psi\colon B(\ell^p, \mathbb{R}) \to \ell^q$ be given by

$$\psi(\gamma) = (\gamma(e_1), \gamma(e_2), \ldots), \qquad \text{where } e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$$

with 1 in the ith spot.

To show: (ba) $\varphi \circ \psi = \text{id}$.
(bb) $\psi \circ \varphi = \text{id}$.

(ba) To show: If $\gamma \in B(\ell^p, \mathbb{R})$ then $\varphi(\psi(\gamma)) = \gamma$.
Assume $\gamma \in B(\ell^p, \mathbb{R})$.
To show: $\varphi(\psi(\gamma)) = \gamma$.
To show: If $x \in \ell^p$ then $\varphi(\psi(\gamma))(x) = \gamma(x)$.
Assume $x \in \ell^p$. Let $x = (x_1, x_2, \ldots)$.
To show: $\varphi(\psi(\gamma))(x) = \gamma(x)$.

$$\begin{aligned}\varphi(\psi(\gamma))(x) &= \varphi(\gamma(e_1), \gamma(e_2), \ldots)(x) = \langle(\gamma(e_1), \gamma(e_2), \ldots), (x_1, x_2, \ldots)\rangle \\ &= \sum_{i\in\mathbb{Z}_{>0}} \gamma(e_i)x_i = \gamma\left(\sum_{i\in ZZ_{>0}} x_i e_i\right) = \gamma(x)\end{aligned}$$

(bb) To show: $\psi \circ \varphi = \text{id}$.
To show: If $y \in \ell^q$ then $\psi(\varphi(y)) = y$.
Assume $y \in \ell^q$. Let $y = (y_1, y_2, \ldots)$.
To show: $\psi(\varphi(y)) = y$.

$$\psi(\varphi(y)) = \psi(\varphi_y) = (\varphi_y(e_1), \varphi_y(e_2), \ldots) = (y_1, y_2, \ldots) = y,$$

since $\varphi_y(e_i) = \langle y, e_i\rangle = \langle(y_1, y_2, \ldots), (0, \ldots, 0, 1, 0, \ldots)\rangle = y_i$.
So $\psi(\varphi(y)) = y$.

(c) To show: If $y \in \ell^q$ then $\|\varphi_y\| = \|y\|_q$.
Assume $y \in \ell^q$. Let $y = (y_1, y_2, \ldots)$.
To show: (ca) $\|\varphi_y\| \le \|y\|_q$.
(cb) $\|\varphi_y\| \ge \|y\|_q$.

(ca) To show: If $x \in \ell^p$ then $|\varphi_y(x)| \le \|x\|_p\|y\|_q$.
Assume $x \in \ell^p$. Let $x = (x_1, x_2, \ldots)$.
Then, by Hölder's inequality,

$$|\varphi_y(x)| = \left|\sum_{n\in\mathbb{Z}_{>0}} x_n y_n\right| \le \|x\|_p\|y\|_q,$$

So $\|\varphi_y\| \le \|y\|_q$.
(cb) To show: $\|\varphi_y\| \ge \|y\|_q$.
To show: There exists $x \in \ell^p$ with $|\varphi_y(x)| \ge \|x\|_p\|y\|_q$.
Let

$$x = \big(\text{sgn}(y_1)|y_1|^{q-1}, \text{sgn}(y_2)|y_2|^{q-1}, \ldots\big).$$

Then

$$\begin{aligned}
\|x\|_p &= \Big(\sum_{n\in\mathbb{Z}_{>0}} |x_n|^p\Big)^{1/p} = \Big(\sum_{n\in\mathbb{Z}_{>0}} |\mathrm{sgn}(y_n)|y_n|^{q-1}|^p\Big)^{1/p} \\
&= \Big(\sum_{n\in\mathbb{Z}_{>0}} |y_n|^{pq-p}\Big)^{1/p} = \Big(\sum_{n\in\mathbb{Z}_{>0}} |y_n|^{pq(1-\frac{1}{q})}\Big)^{1/p} \\
&= \Big(\sum_{n\in\mathbb{Z}_{>0}} |y_n|^{pq\frac{1}{p}}\Big)^{1/p} = \Big(\Big(\sum_{n\in\mathbb{Z}_{>0}} |y_n|^q\Big)^{1/q}\Big)^{q\frac{1}{p}} \\
&= \|y\|_q^{q\frac{1}{p}} = \|y\|_q^{q(1-\frac{1}{q})} = \|y\|_q^{q-1}.
\end{aligned}$$

So

$$\begin{aligned}
|\varphi_y(x)| &= \Big|\sum_{n\in\mathbb{Z}_{>0}} x_n y_n\Big| = \Big|\sum_{n\in\mathbb{Z}_{>0}} \big(\mathrm{sgn}(y_n)|y_n|\big)\big(\mathrm{sgn}(y_n)|y_n|^{q-1}\big)\Big| \\
&= \sum_{n\in\mathbb{Z}_{>0}} |y_n|^q = \|y\|_q^q = \|y\|_q\|y\|_q^{q-1} = \|y\|_q\|x\|_p.
\end{aligned}$$

So $\|\varphi_y\| \geq \|y\|_q$.

□