Lecture 1: Vocabulary

PROOF MACHINE

A Banach space is a complete normed vector space.

A normed vector space is a vector space V with a function $\| \ \|: V \to \mathbb{R}_{\geq 0}$ such that

(a) If $x, y \in V$ then $\|x+y\| \leq \|x\| + \|y\|$,

(b) If $c \in \mathbb{K}$ and $v \in V$ then $\|cv\| = |c|\|v\|$

(c) If $v \in V$ and $\|v\| = 0$ then $v = 0$.

Example $V = \mathbb{R}^2$ is a normed $\mathbb{R}$-vector space with $\| \ \|: V \to \mathbb{R}_{\geq 0}$ given by

$$\|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2}$$

The tolerance set is

$$\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}$$

Define

$$d: V \times V \to \mathbb{R}_{\geq 0} \quad \text{by} \quad d(x,y) = \|y - x\|.$$

For $\varepsilon \in \mathbb{E}$ define

$$B_\varepsilon(x) = \{y \in V \mid d(y,x) < \varepsilon\} \qquad \underline{\varepsilon\text{-ball at } x}$$

$$B_\varepsilon = \{(x,y) \in V \times V \mid d(y,x) < \varepsilon\}. \qquad \underline{\varepsilon\text{-diagonal}}$$

A <u>Cauchy sequence</u> in V is a sequence $(x_1, x_2, \ldots)$ in V such that

if $\varepsilon \in \mathbb{E}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{\geq N}$ then $(x_m, x_n) \in B_\varepsilon$.

A sequence $(x_1, x_2, \ldots)$ <u>converges</u> in V if there exists $v \in V$ such that

if $\varepsilon \in \mathbb{E}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{\geq N}$ then ~~there exists~~ $x_n \in B_\varepsilon(v)$.

In this case write $\lim\limits_{n \to \infty} x_n = v$.

Let $(V, \|\ \|)$ be a normed vector space.
The normed vector space $(V, \|\ \|)$ is <u>complete</u> if $(V, \|\ \|)$ satisfies

if $(x_1, x_2, \ldots)$ is a Cauchy sequence in V then $(x_1, x_2, \ldots)$ converges in V.

In English: complete means every Cauchy sequence converges.

Let $(V, \|\ \|_V)$ and $(W, \|\ \|_W)$ be normed vector spaces and

$d_V : V \times V \to \mathbb{R}_{\geq 0}$ given by $\|v_1 - v_2\| = d_V(v_1, v_2)$

$d_W : W \times W \to \mathbb{R}_{\geq 0}$ given by $d_W(w_1, w_2) = \|w_1 - w_2\|$.

A function $f : V \to W$ is <u>continuous</u> if f satisfies:

if $x \in V$ and $\varepsilon \in \mathbb{E}$ then
there exists $\delta \in \mathbb{E}$ such that
if $y \in V$ and $d_V(x, y) < \delta$ then $d_W(f(x), f(y)) < \varepsilon$

A function $f : V \to W$ is <u>uniformly continuous</u> if f satisfies

if $\varepsilon \in \mathbb{E}$ then
there exists $\delta \in \mathbb{E}$ such that
if $x \in V$ and $y \in V$ and $d_V(x, y) < \delta$
then $d_W(f(x), f(y)) < \varepsilon$.